Lecture notes on

Boundary value problems with nonsmooth and multivalued terms*

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1 Introduction

In these lecture notes we introduce some actual methods and tools used in the study of nonlinear boundary value problems and illustrate them in our recent results, concerning existence and multiplicity of solutions to some Dirichlet and Neumann problems for nonlinear elliptic equations driven by the p–Laplacian differential operator with nonsmooth and multivalued terms.

The paper is organized as follows. In Section 2 we recall some notions and facts concerning the calculus of smooth functions and introduce some notions of nonsmooth analysis, that we will use in the next sections.

The critical point theory for smooth and for nonsmooth functionals developed in Section 3, is the main tool in the variational method for solving boundary value problems.

The need of specific applications such as nonsmooth mechanics, nonsmooth gradient systems, mathematical economics, etc.) and the impreesive progress in nonsmooth analysis and multivalued analysis led to extensions of the critical point theory to nondifferentiable functions, in particular locally Lipschitz and even continuous functions.

The resulting *nonsmooth critical point theory* succedded in extending a big part of the smooth (C^1) theory. We present the main facts of this theory that we used to solve certain boundary value problems for nonlinear ordinary and elliptic partial differential equations and we refer to [27] for further results and details.

The variational method studied in Section 4 consists of trying to find solutions for a given boundary value problem by looking for stationary points of a real functional defined on a space of functions in which the solution of the boundary value problem is to lie. we discuss the variational method and illustrate it in recent results, where both smooth and the nonsmooth critical point theory is used. We consider Dirichlet and Neumann problems for nonlinear elliptic equations driven by the ordinary and partial p–Laplacian differential operator. We used this method to obtain existence of multiple solutions to nonlinear Dirichlet problem with nonsmooth potential (hemivariational inequality) [45] and also to prove existence of solutions with precise sign information for the p-Laplacian Neumann problem [3], existence of multiple constant sign solutions and of nodal solutions when the nonlinear perturbation have superlinear growth near infinity [4].

Section 5 is dedicated to the spectrum of the negative p-Laplacian both for Dirichlet boundary conditions and for Neumann boundary conditions. We recall first important known results starting with the ordinary ($N = 1$) case, and then we recall some basic results for $N \geq 1$. In the last part of this section we emphasize some recent results obtained in [4] concerning the spectrum of the negative $p-$ Laplacian with Neumann boundary conditions (denoted by $(-\triangle_p, W^{1,p}(Z))$: an alternative variational characterizations of the first nonzero eigenvalue of the negative $p-$ Laplacian with Neumann boundary conditions, distinct from the one determined by the Lusternik-Schnirelmann theory; the continuous dependence of the eigenvalues on $p \in (1, +\infty)$; the isolation of the principal eigenvalue

$\lambda_0 = 0$ is uniform for all p in a bounded interval, and an index formula (jumping theorem) for the degree of the nonlinear operator corresponding to the eigenvalue problem, as we approach the eigenvalue from above and below. Those results have been applied in [**?**] to prove a multiplicity result for nonlinear Neumann problems with a multivalued crossing nonlinearity.

In Section 6 we present the method of upper and lower solutions, which provides an effective tool to produce existence theorems for first and second order initial and boundary value problems and to generate monotone iterative techniques which provide constructive methods (amenable to numerical treatment), to obtain solutions. We apply this method combined with some variational arguments to prove multiplicity results to some perturbed eigenvalue problems with the $p-$ Laplacian and a nonlinear perturbation. In addition, we also establish the existence of extremal solutions in the order interval formed by an ordered pair of upper and lower solutions.

Section 7 is dedicated to degree theory, which is a basic tool of nonlinear analysis and produces powerful existence and multiplicity results for nonlinear boundary value problems. A special atention is dedicated to a generalization of Brouwer degree theory to multivalued perturbations of monotone type maps, developed in our joint paper [2].

In Section 8 we illustrate the degree theoretical approach in the study of nonlinear boundary value problems. Recent generalizations of degree theory to nonlinear operators of monotone type, paved the way to use degree theoretical methods to boundary value problems of nonlinear partial differential equations and to problems with unilateral constraints (such as variational and hemivariational inequalities).

In Section 9 we recall some basic notions and results from Morse theory, which we will need in Section 10 produce nontrivial smooth solutions and obtain multiplicity results.

Several of the quoted results use a combinations of the methods from above: variational com some truncations, degree theoretic approach and/or upper-lower solutions method.

In Section 11 we refer to some complementary topics of set-valued analysis, differential inclusions and their relations with control theory and variational calculus,

2 Smooth and nonsmooth calculus

In this section we present some basic aspects of smooth and nonsmooth calculus in Banach spaces, providing some basic tools to approach the subjects in the sections that follows.

Let X be a Banach space, X^* its topological dual and by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*) (the pairing between X^* and X). Let φ be a real function defined in a nonempty open subset $U \subseteq X$. If we want to find necessary condition for f have a minimum, we need a suitable generalization of the concept of derivative. The simplest one is the Gateaux derivative.

Definition 1 *We say that $\varphi : U \subseteq X \to \mathbb{R}$ has a Gateaux derivative $x^* \in X^*$ at $x \in U$ if, for every $h \in X$ one has*

$$\limsup_{t\to 0} \frac{\varphi(x+th) - \varphi(x) - \langle x^*, th\rangle}{t} = 0.$$

We denote x^* by $\varphi'(x)$.

A stronger form of diferentiability is Fréchet diferentiability

Definition 2 *We say that $\varphi : U \subseteq X \to \mathbb{R}$ has a Fréchet derivative $x^* \in X^*$ at $x \in U$ if*

$$\limsup_{h\to 0} \frac{\varphi(x+h) - \varphi(x) - \langle x^*, h\rangle}{t} = 0.$$

Of course the Fréchet derivative at x is the Gateux derivative at x and we keep the notation $\varphi'(x)$.

We say that $\varphi \in C^1(U, \mathbb{R})$ or that φ is *continuously Fréchet differentiable*) if the Fréchet derivative $\varphi'(x)$ exists at each $x \in U$ and the map $x \longrightarrow \varphi'(x)$ is continuous from X into X^*. We denote by $C^1(U)$ the set of all real continuous differentiable functions defined on U.

With those definitions, it is straightforward to generalize to the setting of Banach spaces Fermat's necessary condition for the existence of a local minimum or maximum of φ at $x \in U$,

$$\varphi'(x) = 0, \tag{1}$$

when φ is Gateux differentiable at $x \in U$.

Any $x \in U$ satisfying (1) is called a *critical point* of φ and $\varphi(x)$ is called a *critical value.*

Consequently, if a mapping $\Phi : X \to X^*$ can be written as $\Phi = \varphi'$ for some Gateaux differentiable function $\varphi : X \to \mathbb{R}$, every critical point of φ provide a solution of the equation

$$\Phi(u) = 0.$$

It is in particular the case for any local minimum and local maximum of φ.

Let now $\varphi \colon X \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. We say φ is a *proper function* if its *effective domain*

$$dom(\varphi) := \{x \in X \colon \varphi(x) < +\infty\}$$

is nonempty. We say that φ is a *convex* function if for all $x_1, x_2 \in dom(\varphi)$ and all $\lambda \in [0,1]$, we have

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2).$$

If this inequality is strict when $x_1 \neq x_2$ and $\lambda \in (0,1)$, then we say that φ is *strictly convex.*

It is easy to check that φ is convex if and only if its epigraph

$$epi(\varphi) = \{(x, \lambda) \in X \times \mathbb{R} \colon \varphi(x) \leq \lambda\}$$

is convex. We say that $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is *lower semicontinuous* if for every $\lambda \in \mathbb{R}$ the sublevel set

$$L_\lambda(\varphi) = \{x \in X\colon \varphi(x) \leq \lambda\}$$

is closed. It is well known that a convex and lower semicontinuous function $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is continuous on the interior of *dom* (φ).

We denote by $\Gamma_0(X)$, the set of all proper, convex and lower semicontinuous functions from X into $\mathbb{R}$.

The subdifferential of a convex function characterizes the local behavior of the convex function in a way analogous to that in which derivatives determine the local behavior of smooth functions.

Let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper function and $x_0 \in$ *dom* (φ). The *subdifferential (in the sense of convex analysis)* of φ at x_0 is the set $\partial_c\varphi(x_0) \subseteq X^*$ (possibly empty) defined by

$$\partial_c\varphi(x_0) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi(x_0 + h) - \varphi(x_0) \text{ for } h \in X\}. \tag{2}$$

The generalization of Fermat's necessary condition for the case of convex functions is the following: $x_0 \in X$ is a minimum of the convex function $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ if and only if

$$0 \in \partial\varphi(x_0).$$

If X is a Banach space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper convex function which is Gâteaux differentiable at x_0 then

$$\partial\varphi(x_0) = \{\varphi\prime(x_0)\}.$$

The nonsmooth critical point theory that we will study in Section 3 and which we employ in the variational arguments, is based mainly on the subdifferential theory for locally Lipschitz functions. Further details on these subjects can be found in the books of Clarke [18] (the subdifferential theory) and Gasinski-Papageorgiou [27] (the nonsmooth critical point theory).

Let, as before, X be a Banach space and X^* be its dual and let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. The *generalized directional derivative* $\varphi^0(x; h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}. \tag{3}$$

It is straightforward to check that $h \to \varphi^0(x; h)$ is continuous, sublinear and so it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \left\{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \text{ for all } h \in X\right\}. \tag{4}$$

The multifunction $x \to \partial\varphi(x)$ is called the (Clarke) *generalized subdifferential* of φ.

If $\varphi \in C^1(X)$, then φ is locally Lipschitz and

$$\partial\varphi(x) = \{\varphi'(x)\}.$$

Also, if $\varphi : X \to \mathbb{R}$ is a continuous convex function, then the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis $\partial_c\varphi(x)$ defined by (2). If $\varphi,\, \psi : X \to \mathbb{R}$ are locally Lipschitz functions, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \text{ for all } x \in X.$$

The extension to the present nonsmooth setting of the Fermat's necessary condition for the existence of local extrema is the following:

Proposition 3 *If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz functions which attains a local extrema (local minimum or local maximum) at $x \in X$ then*

$$0 \in \partial\varphi(x). \tag{5}$$

3 Smooth and nonsmooth critical point theory

In this section we develop the critical point theory for smooth and nonsmooth functionals, the main tools in the variational method. This method consists of trying to find solutions of a given equation, by looking for stationary points of a real functional $\varphi(.)$ defined on the function space X in which the solution of the equation is to lie. The given equation is the *Euler–Lagrange equation* satisfied by a *stationary point.*

To guaranty minimizers (or maximizers) we need two types of properties of the functional $\varphi(.)$ we need two types of properties of the functional φ : one is *quantitative* and requires that the sublevel sets $L_\lambda(\varphi) = \{x \in X \colon \varphi(x) \leq \lambda\}$ are relatively compact in some useful topology on X (coercivity property) and the other *qualitative,* which requires the lowes semicontinuity (or upper semicontinuity) of $\varphi(.)$ for the same topology on X.

A coercive functional is bounded below. But many functionals which we may encounter often, may be not bounded at all, nether above or from below.

Thus we need results where other types of critical points may be identified.

Definition 4 *Let X be a Banach space and let $\varphi \in C^1(X)$. We sau that $x \in X$ is a* critical point *of φ, if $\varphi'(x) = 0$. We say that $c \in \mathbb{R}$ is a* critical value *of φ if there is a critical point $x \in X$ such that $c = \varphi(x)$. We say that $c \in \mathbb{R}$ is a* regular value *of φ if it is not a critical value of φ.*

Let X be a Banach space and $\varphi \in C^1(X)$. For every $\lambda \in \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and every $c \in \mathbb{R}$, we introduce the following notations:

$$\begin{aligned} \varphi^\lambda &= \{x \in X : \varphi(x) \leq c\}, \\ K &= \{x \in X : \varphi'(x) = 0\} \\ K_c &= \{x \in K : \varphi(x) = c\}, \end{aligned}$$

for the sublevel set of φ *at* λ, the *critical set of* φ and the *critical set of* φ *at level* $c \in \mathbb{R}$, respectively.

In order to get informations about the critical set K we employ the so-called *deformation method*.

Definition 5 *A continuous function* $h : [0,1] \times X \to X$ *is called a* deformation *of* X *if* $h(0,x) = x$ *for all* $x \in X$. *A family* $\mathcal{S}$ *of subsets of* X *is said to be* deformation invariant *if* $h(1,A) \in \mathcal{S}$ *for every* $A \in \mathcal{S}$ *and every* $h(.,.)$ *deformation of* X.

We are interested on deformations $h(.,.)$ which effectively decrease the values of φ on $X \backslash K$ and try to determine critical values of φ characterized by minimax expresion of the form

$$c = \inf_{A \in \mathcal{S}} \sup_{x \in A} \varphi(x)$$

for various deformation invariant classes $\mathcal{S}$. The construction of suitable deformations is the most technical part of the deformation method, and a guiding condition to produce apropriate deformations is the following

Condition (D): For $h : [0,1] \times X \to X$ a deformation of X, we have:

(i) for all $-\infty < a < b < +\infty$ with $\varphi^{-1}([a,b]) \cap K = \varnothing$, there exists $t_0 > 0$ such that

$$h\left(t_0, \varphi^b\right) \subseteq \varphi^a;$$

(ii) if $c \in \mathbb{R}$ and U is a neighborhood of K_c, there exists $t_0 > 0$ and $a, b \in \mathbb{R}$ such that $a < c < b$ for which we have

$$h\left(t_0, \varphi^b\right) \subseteq U \cup \varphi^a.$$

Remark 6 *Condition* (i) *says that* $h(.,.)$ *effectively decreases the values of* φ *in* $X \backslash K$. *So nothing can happen topologically between the levels* a *and* b, *if the interval* $[a,b]$ *does not contains any critical value. On the other hand* (ii) *says that if we start a little above a critical level* c, *then we either bypass the critical neighborhood* U *and as before reach a harmless level* $a < c$ *or will end up in* U, *where topologically interesting things may happen.*

The deformation theorem that we will prove in the sequel, shows how to construct deformations satisfying condition (D) above. Let introduce now the following compactness-type conditions

Definition 7 *Let* X *be a Banach space with norm* $\|.\|$, *and let* $\varphi \in C^1(X)$.

We say that φ *satisfies the* Palais–Smale condition *at level* $c \in R$ *(the* PS_c–condition *for short), if every sequence* $(x_n)_{n \geq 1} \subseteq X$ *such that*

$$\varphi(x_n) \longrightarrow c \text{ and } \varphi\prime(x_n) \longrightarrow 0,$$

has a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, *then we simply say that* φ *satisfies the* PS*–condition.*

We say that φ satisfies the Cerami condition *at level $c \in \mathbb{R}$ (the C_c–condition for short), if every sequence $(x_n)_{n\geq 1} \subseteq X$ such that*

$$\varphi(x_n) \longrightarrow c \text{ and } (1 + \|x_n\|)\, \varphi\prime(x_n) \longrightarrow 0,$$

has a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, then we simply say that φ satisfies the C–condition.

Clearly C_c–condition is weaker that PS_c–condition. If X is a Banach space and $\varphi \in C^1(X)$ is bounded below and satisfies $C-$ condition then φ is weakly coercive, i.e.,

$$\varphi(x) \longrightarrow +\infty \text{ as } \|x\| \longrightarrow +\infty.$$

Moreover, if X is a Banach space and $\varphi \in C^1(X)$ is bounded below then PS–condition and C–condition are equivalent.

The next result is known as Deformation Theorem which will lead to minimax characterizations of the critical values of φ (for the proof see [27], p. 132).

Theorem 8 *If X is a Banach space and $\varphi \in C^1(X)$ satisfies the C_c–conditionfor some $c \in \mathbb{R}$ then for every $\varepsilon_0 > 0$ and every neighborhood U of K_c (if $K_c = \varnothing$, then $U = \varnothing$) and every $\lambda > 0$,we can find $\varepsilon \in (0, \varepsilon_0)$ and a continuous map $h : [0,1] \times X \longrightarrow X$ (a continuous homotopy) such that for all $(t,x) \in [0,1] \times X$,we have:*

(i) $\|h(t,x) - x\| \leq \lambda(1 + \|x\|)t$;
(ii) $\varphi(h(t,x)) \leq \varphi(x)$;
(iii) $h(t,x) \neq x$ *implies* $\varphi(h(t,x)) < \varphi(x)$;
(iv) $|\varphi(x) - c| \geq \varepsilon_0$ *implies* $h(t,x) = x$;
(v) $h(1, \varphi^{c+\varepsilon}) \subseteq \varphi^{c-\varepsilon} \cup U$.

Now we will use this deformation theorem to produce minimax characterizations of the critical values of $\varphi \in C^1(X)$. For this we will need the following basic topological notion:

Definition 9 *Let Y be a Hausdorff topological space, $E_0 \subseteq E$ and D are nonempty subsets of Y with D closed and $\gamma^* \in C(E_0, X)$. We say that the sets $\{E_0, E\}$ and D* link *in Y via γ^* if and only if the following conditions are both satisfied:*

(a) $E_0 \cap D = \varnothing$
(b) for any $\gamma \in C(E,X)$ such that $\gamma|_{E_0} = \gamma^*|_{E_0}$, we have $\gamma(E) \cap D \neq \emptyset$.

The sets $\{E_0, E, D\}$ are said to be linking sets via $\gamma^* \in C(E_0, X)$. *If $\gamma^* = Id|_{E_0}$, then we simply say that $\{E_0, E, D\}$ are* linking sets.

Theorem 10 *If X is a Banach space, $\varphi \in C^1(X)$, the sets $\{E_0, E, D\}$ are linking via γ^*,*

$$\alpha = \sup_{E_0} \varphi < \inf_{D} \varphi = \beta,$$
$$\Gamma = \{\gamma \in C(E,X) : \gamma|_{E_0} = \gamma^*\},$$
$$c = \inf_{\gamma\in\Gamma} \sup_{x\in E} \varphi(\gamma(x))$$

and φ satisfies the C_c–condition, then $c \geq \beta$ and c is a critical value of φ.

Proof. Since by hypothesis the sets $\{E_0, E, D\}$ are linking via $\gamma^* \in C(E_0, X)$, for every $\gamma \in \Gamma$ we have $\gamma(E) \cap D \neq \emptyset$ and so $\beta \leq c$.

To show that c is a critical value of φ, we argue by contradiction. So suppose by that $K_c = \emptyset$. Set $\varepsilon_0 = \beta - \alpha$ and $U = \varnothing$. Then by virtue of the Deformation Theorem, we can find $0 < \varepsilon < \varepsilon_0$ and a continuous homotopy $h\colon [0,1] \times X \longrightarrow X$ which satisfies statements $(i) - (v)$ of that theorem. We choose $\gamma \in \Gamma$ such that

$$\varphi(\gamma\,(x)) < c + \varepsilon \text{ for all } x \in E. \tag{6}$$

Set $\gamma_0\,(x) = h\,(1, \gamma_0\,(x))$ for all $x \in E$. From the choice of $\varepsilon_0 > 0$ and statement (iv) in Deformation Theorem we see that $\gamma_0|_{E_0} = \gamma^*$ and so $\gamma_0 \in \Gamma$. For all $x \in E$, we have

$$\varphi(\gamma_0\,(x)) = \varphi\,(h\,(1, \gamma\,(x)))\,. \tag{7}$$

Combining (6) and (7) with statement (v) of deformation theorem we infer that

$$\varphi(\gamma_0\,(x)) \leq c - \varepsilon \text{ for all } x \in E,$$

a contradiction to the definition of c. ■

Now with suitable choices of the linking sets, we can have the *Mountain pass theorem* and the *Saddle point theorem*.

We start with the *Mountain pass theorem*:

Theorem 11 *If X is a Banach space, $\varphi \in C^1(X)$, $x_0, x_1 \in X$ and $r > 0$ are such that $\|x_1 - x_0\| \geq r$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi\,(x) : \|x - x_0\| = r\},$$

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi\,((\gamma(t))$$

and φ satisfies the C_c–condition, then

$$c \geq \inf\{\varphi : \|x - x_0\| = r\}$$

and c is a critical value of φ.

Proof. We consider the linking sets $E_0 = \{x_0, x_1\}$, $E = \{(1-t)x_0 + tx_1 : t \in [0,1]\}$, $D{=}\partial B_r(x_0)$ and we apply Theorem 11. ■

Next we state the *Saddle point theorem*:

Theorem 12 *If X is a Banach space, $X = Y \oplus V$ with $\dim Y < +\infty$, $\varphi \in C^1(X)$, there exists $R > 0$ such that*

$$\max\{\varphi(x) : x \in \partial B_R(0) \cap Y\} < \inf\{\varphi(x) : x \in V\} = \beta$$

$$\Gamma = \{\gamma \in C\left(\overline{B}_R(0) \cap Y, X\right) : \gamma|_{\partial \overline{B}_R(0)\cap Y} = Id_{\partial \overline{B}_R(0)\cap Y}$$

$$c = \inf_{\gamma\in\Gamma} \sup_{x\in E} \varphi\,(\gamma(x))$$

and φ satisfies the C_c–condition, then $c \geq \beta$ and c is a critical value of φ.

Proof. We consider the linking sets $E_0 = \partial B_R(0) \cap Y,\ E = \overline{B}_R(0) \cap Y,\ D = V$ and we apply Theorem 11. ∎

Next we look for multiple critical points of a smooth functional φ. For this purpose we introduce the following notion.

Definition 13 *Let X be a Banach space, $X = Y \oplus V$ and $\varphi \in C^1(X)$. We say that φ has a* local linking *at* 0*, if there exists $r > 0$ such that*

$$\begin{cases} \varphi(x) \leq 0 & \text{if } x \in Y,\ \|x\| \leq r \\ \varphi(x) \geq 0 & \text{if } x \in V,\ \|x\| \leq r. \end{cases}$$

Remark that $x = 0$ is a critical point of φ. The next result due to Brezis-Nirenberg [12] furnishes two more, distinct nontrivial critical points of φ.

Theorem 14 *If X is a Banach space, $X = Y \oplus V$ with dim $Y < \infty$, $\varphi \in C^1(X)$ is bounded below, satisfies PS-condition and is such that*

$$m = \inf_{x \in X} \varphi(x) < \varphi(0) = 0$$

and there exists $r > 0$ such that

$$\begin{cases} \varphi(x) \leq 0 & \text{if } x \in Y,\ \|x\| \leq r \\ \varphi(x) \geq 0 & \text{if } x \in V,\ \|x\| \leq r. \end{cases}$$

then φ has at least two nontrivial critical points.

For the proof we refer to [12] and to [28], p. 661)

The nonsmooth critical point theory that we consider in the remaining of this section, will be employed in the variational method and it is based mainly on the subdifferential theory for locally Lipschitz functions (see [18] and [27]).

Let X be a Banach space, X^* be its topological dual and let $\langle \cdot, \cdot \rangle$ be the duality brackets for the pair (X, X^*). Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional.

Let $\varphi^0(x; h)$ be the *generalized directional derivative* of φ at $x \in X$ in the direction $h \in X$ defined by (3) and for each $x \in X$ let $\partial\varphi(x) \subseteq X^*$ be the Clarke subdifferential of φ defined by (4). Recall that $\partial\varphi(x)$ is a nonempty, convex, w*-compact subset of X^* and, if $\varphi \in C^1(X)$ then φ is locally Lipschitz and

$$\partial\varphi(x) = \{\varphi'(x)\}.$$

Definition 15 *We say that $x \in X$ is a* critical point *of a locally Lipschitz function $\varphi : X \to \mathbb{R}$ if*

$$0 \in \partial\varphi(x).$$

If $x \in X$ is a critical point then $c = \varphi(x)$ is called a critical value*. of φ.*

Definition 16 *Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we say that φ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (PS_c-condition, for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that*

$$\varphi(x_n) \to c \text{ and } m(x_n) := \inf [\|x^*\| : x^* \in \partial\varphi(x_n)] \to 0, \text{ as } n \to \infty,$$

has a strongly convergent subsequence. We say that φ satisfies the PS-condition, if it satisfies the PS_c-condition for every $c \in \mathbb{R}$.

Sometimes, it is more appropriate to use a slightly more general compactness notion, the so-called *Cerami condition*.

Definition 17 *We say that a locally Lipschitz function $\varphi : X \to \mathbb{R}$, satisfies the Cerami condition at level $c \in \mathbb{R}$ (C_c-condition, for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that*

$$\varphi(x_n) \to c \text{ and } (1 + \|x_n\|) m(x_n) \to 0, \text{ as } n \to \infty,$$

has a strongly convergent subsequence. We say that φ satisfies the C-condition, if it satisfies the C_c-condition for every $c \in \mathbb{R}$.

Remark that the Cerami condition is weaker than Palais-Smale condition, and the two notions coincide if φ is bounded below.

The topological notion of linking sets is crucial in the minimax characterization of the critical values of a locally Lipschitz function.

Definition 18 *Let Y be a Hausdorff topological space, E_0, E and D are nonempty closed subsets of Y, with $E_0 \subseteq E$. We say that the pair $\{E, E_0\}$ is linking with D in Y, if*

(i) $E_0 \cap D = \varnothing$;

(ii) for any $\gamma \in C(E, Y)$, with $\gamma|_{E_0} = id|_{E_0}$, we have $\gamma(E) \cap D \neq \varnothing$.

Using this topological notion, we get the following general minimax principle for the critical values of a locally Lipschitz function (see Kourogenis-Papageorgiou [33]):

Theorem 19 *If E_0, E and D are nonempty, closed subsets of X, $\{E, E_0\}$ is linking with D in X, $\varphi : X \to \mathbb{R}$ is locally Lipschitz,*

$$\sup_{E_0} \varphi \leq \inf_D \varphi,$$

φ satisfies the C_c-condition, where

$$c = \inf_{\gamma \in \Gamma} \sup_{x \in E} \varphi(\gamma(x)) \text{ and } \Gamma = \{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\}$$

then

$$c \geq \inf_D \varphi$$

and c is a critical value of φ. Moreover, if

$$c = \inf_{D} \varphi,$$

then there exists a critical point x of φ, such that $\varphi(x) = c$ and $x \in E$.

With suitable choices of linking sets, we produce nonsmooth versions of well-known minimax theorems. We mention the *nonsmooth mountain pass theorem*, which we shall need in the sequel.

Theorem 20 *If x_0, $x_1 \in X$ with $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \leq \inf \varphi(x) : \|x\| = r$$

and φ satisfies the PS_c-condition, where

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)), \ \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_{0,} \ \gamma(1) = x_1\}$$

then

$$c \geq \inf_{\|x\|=r} \varphi(x)$$

and c is a critical value of φ, with $\varphi(x) = c$. Moreover, if

$$c = \inf_{\|x\|=r} \varphi(x),$$

then there exists a critical point x of φ, with

$$\varphi(x) = c \text{ and } \|x\| = r.$$

More about the nonsmooth critical point theory can be found in the books of Carl-Le-Motreanu [15], Gasinski-Papageorgiou [27] and Motreanu-Radulescu [42]. Recently, Kandilakis-Kourogenis-Papageorgiou [32] (see also Gasinski-Papageorgiou [27], p.178), extended to a nonsmooth setting the local linking theorem of Brezis-Nirenberg [12].

Theorem 21 *If $X = Y \oplus V$, with $\dim Y < +\infty$, φ is locally Lipschitz on bounded sets, bounded below, satisfies the PS-condition, $\varphi(0) = 0$, $\inf\{\varphi(x) : x \in X\} < 0$ and there exists $r > 0$, such that*

$$\begin{cases} \varphi(x) \leq 0 & \text{if } x \in Y, \ \|x\| \leq r \\ \varphi(x) \geq 0 & \text{if } x \in V, \ \|x\| \leq r. \end{cases}$$

then φ has at least two nontrivial critical points.

4 Variational method

In this section we use the variational method to solve some characteristic second order boundary value probems under Dirichlet and under Neumann boundary conditions.

This method consists of trying to find solutions for a given boundary value problem by looking for stationary points of a real functional defined on a space of functions in which the solution of the boundary value problem is to lie.

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2 boundary ∂Z. We consider the following nonlinear Dirichlet problem with nonsmooth potential (hemivariational inequality) (see [45]),

$$\begin{cases} -div\left(\|Dx(z)\|^{p-2} Dx(z)\right) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ x\,|_{\partial Z} = 0,\ 1 < p < \infty. \end{cases} \tag{8}$$

and we want to apply variational method to obtain two of the three solutions for this problem A third solution will be obtained ater by using a degree theory argument.

The hypotheses on the nonsmooth potential function $j(z, x)$ are the following:

$H(j):$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that for almost all $z \in Z$, all $|x| \le r$ and all $u \in \partial j(z, x)$, we have

$$|u| \le a_r(z);$$

(iv) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \le \lambda_1$ a.e. on Z with strict inequality on a set of positive measure and

$$\limsup_{|x| \to +\infty} \frac{u}{|x|^{p-2} x} \le \theta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z, x)$;

(v) there exist functions $\eta_1, \eta_2 \in L^\infty(Z)_+$ with $\lambda_1 \le \eta_1(z) \le \eta_2(z)$ a.e. on Z, the first inequality is strict on a set of positive measure,

$$\eta_1(z) \le \liminf_{x \to 0^+} \frac{u}{x^{p-1}} \le \limsup_{x \to 0^+} \frac{u}{x^{p-1}} \le \eta_2(z)$$

and

$$\lim_{x \to 0^-} \frac{u}{|x|^{p-2} x} = 0,$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z, x)$;

(vi) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$ we have

$$ux \geq 0. \qquad \text{(sign condition)}$$

Remark 22 *Note that at 0^+ and $\pm\infty$, we allow partial interaction with the principal eigenvalue $\lambda_1 > 0$ (nonuniform nonresonance). When $p = 2$ (semilinear problems), hypotheses $H(j)$ incorporate in our framework of analysis, the so-called asymptotically linear problems, which attracted considerable interest since the appearance of the pioneering work of Amann-Zehnder [5]. We point out, that the hypotheses are asymmetric with respect to 0^+ and 0^-. Moreover, it is worth mentioning that as we move from 0^+ to $+\infty$, we cross the principal eigenvalue $\lambda_1 > 0$.*

The following simple nonsmooth, locally Lipschitz function satisfies all hypotheses $H(j)$. For simplicity we drop the z dependence in its definition:

$$j(x) = \begin{cases} \frac{c}{p}|x|^p + \frac{1}{r} - \frac{c}{p} & \text{if } x < -1 \\ \frac{1}{r}|x|^r & \text{if } x \in [-1,0] \\ \frac{\eta}{p}x^p & \text{if } x \in [0,1] \\ \frac{\theta}{p}x^p + \frac{1}{p}\ln x^p + \frac{\eta-\theta}{p} & \text{if } x > 1 \end{cases},$$

with $1 < p < r < \infty$ and $c, \theta < \lambda_1 < \eta$.

In this section, employing a variational approach based on nonsmooth analysis, we establish the existence of two smooth constant sign solutions for problem (8). To this end, we need to truncate the potential function and consider the energy functional corresponding to the truncated potential. So let $\tau_\pm : \mathbb{R} \to \mathbb{R}_+$ be the Lipschitz continuous truncation maps defined by

$$\tau_+(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} \quad \text{and } \tau_-(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}.$$

We set

$$j_\pm(z,x) = j(z, \tau_\pm(x)).$$

Clearly for every $x \in \mathbb{R}$, $z \to j_\pm(z,x)$ are measurable and for almost all $z \in Z$, $x \to j_\pm(z,x)$ are locally Lipschitz. Note that since by hypothesis $j(z,0) = 0$ a.e. on Z, for almost all $z \in Z$ and all $x \leq 0$ (resp. $x \geq 0$) we have $j_+(z,x) = 0$ (resp. $j_-(z,x) = 0$). Moreover, from the nonsmooth chain rule (see Clarke [**?**], p.42), we have

$$\partial j_+(z,x) \subseteq \begin{cases} \{0\} & \text{if } x < 0 \\ \{r\partial j(z,0) : r \in [0,1]\} & \text{if } x = 0 \\ \partial j(z,x) & \text{if } x > 0 \end{cases} \tag{9}$$

and

$$\partial j_-(z,x) \subseteq \begin{cases} \partial j(z,x) & \text{if } x < 0 \\ \{r\partial j(z,0) : r \in [0,1]\} & \text{if } x = 0 \\ \{0\} & \text{if } x > 0 \end{cases}. \tag{10}$$

We consider the functionals $\varphi_{\pm} : W_0^{1,p}(Z) \to \mathbb{R}_+$ defined by

$$\varphi_{\pm}(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_{\pm}(z, x(z))\, dz \text{ for all } x \in W_0^{1,p}(Z).$$

Then $\varphi_{\pm}$ are Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [**?**], p.83). In what follows, for notational simplicity we set

$$W_+ = W_0^{1,p}(Z)_+ = \left\{x \in W_0^{1,p}(Z) : x(z) \geq 0 \text{ a.e. on } Z\right\}$$

$$C_+ = C_0^1\left(\overline{Z}\right)_+ = \left\{x \in C_0^1\left(\overline{Z}\right)_+ : x(z) \geq 0 \text{ for all } z \in Z\right\}$$

and

$$C_- = -C_+.$$

As we already mentioned in Section 2, *int* $C_+ \neq \varnothing$ and *int* C_+ is given by (52). Hereafter, by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $\left(W^{-1,p'}(Z) = W_0^{1,p}(Z)^*, W_0^{1,p}(Z)\right)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We consider the nonlinear operator $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$\langle A(x), y \rangle = \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N}\, dz \text{ for all } x,\, y \in W_0^{1,p}(Z). \tag{11}$$

It is straightforward to check that A is bounded, continuous, monotone, hence it is maximal monotone. Also let $N_{\pm} : L^p(Z) \to 2^{L^{p'}(Z)}$ be the multifunctions defined by

$$N_{\pm}(x) = \left\{u \in L^{p'}(Z) : u(z) \in \partial j_{\pm}(z, x(z)) \text{ a.e. on } Z\right\} \tag{12}$$

for all $x \in W_0^{1,p}(Z)$. These are the multivalued Nemytskii operators corresponding to the subdifferentials $x \to \partial j_{\pm}(z, x)$. We have

$$\partial \varphi_{\pm}(x) = A(x) - N_{\pm}(x) \text{ for all } x \in W_0^{1,p}(Z). \tag{13}$$

The next proposition is crucial in obtaining the constant sign solutions of problem (8). It underlines the significance of the nonuniform nonresonance condition at $\pm\infty$ (hypothesis $H(j)$ (iv)) and implies that the functionals $\varphi_{\pm}$ are coercive. This fact makes possible the use of variational techniques.

Proposition 23 *If $\theta \in L^{\infty}(Z)_+$ satisfies $\theta(z) \leq \lambda_1$ a.e. on Z with strict inequality on a set of positive measure, then there exists $\xi_0 > 0$ such that*

$$\psi(x) = \|Dx\|_p^p - \int_Z \theta(z) |x(z)|^p\, dz \geq \xi_0 \|Dx\|_p^p \text{ for all } x \in W_0^{1,p}(Z). \tag{14}$$

Proof. From the variational characterization of λ_1 (see the previous proposition and (49)) we have that $\psi \geq 0$. Suppose that (14) is not true. Since ψ is p-positively homogeneous, we can find $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that

$$\|x_n\| = 1 \text{ for all } n \geq 1 \text{ and } \psi(x_n) \downarrow 0 \text{ as } n \to \infty.$$

We may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z), \; x_n \to x \text{ in } L^p(Z), \; x_n(z) \to x(z) \text{ a.e. on } Z$$

and

$$|x_n(z)| \leq k(z) \text{ a.e. on } Z, \text{ for all } n \geq 1 \text{ with } k \in L^p(Z)_+ .$$

We have $Dx_n \xrightarrow{w} Dx$ in $L^p\left(Z, \mathbb{R}^N\right)$ and so

$$\|Dx\|_p^p \leq \liminf_{n\to\infty} \|Dx_n\|_p^p . \tag{15}$$

Also from the dominated convergence theorem, we have

$$\int_Z \theta(z) |x_n(z)|^p \, dz \to \int_Z \theta(z) |x(z)|^p \, dz \text{ as } n \to \infty. \tag{16}$$

From (15) and (16), we have

$$\psi(x) \leq \lim_{n\to\infty} \psi(x_n) = 0,$$

hence

$$\|Dx\|_p^p \leq \int_Z \theta(z) |x(z)|^p \, dz \leq \lambda_1 \|x\|_p^p . \tag{17}$$

Because of (49) and since the infimum in that expression is attained at u_1, from (17) it follows that $x = 0$ or $x = \pm u_1$.

If $x = 0$, then $\|Dx_n\|_p \to 0$ and so by Poincaré's inequality $x_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \geq 1$.

If $x = \pm u_1$, then from the first inequality in (17) and since $|x(z)| = u_1(z) > 0$ for all $z \in Z$, due to the hypothesis on θ, we obtain

$$\|Dx\|_p^p < \lambda_1 \|x\|_p^p ,$$

a contradiction to (49). This proves (14). ■

Using this proposition and a variational argument based on notions from nonsmooth analysis, we can produce the first two solutions of constant sign for problem (8).

Theorem 24 *If hypotheses $H(j)$ hold, then problem* (8) *has two solutions $x_0 \in$ int C_+ and $v_0 \in -$int C_+.*

Proof. By virtue of hypothesis $H(j)$ (iv), given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that for almost all $z \in Z$, all $x \geq M$ and all $u \in \partial j(z,x) = \partial j_+(z,x)$, we have

$$u \leq (\theta(z) + \varepsilon) x^{p-1}. \tag{18}$$

On the other hand, hypothesis $H(j)$ (iii) and (9) imply that there exists $a_\varepsilon \in L^\infty(Z)_+$ such that for almost all $z \in Z$, all $0 \leq x < M$ and all $u \in \partial j_+(z,x)$, we have

$$u \leq a_\varepsilon(z). \tag{19}$$

Finally note that for almost all $z \in Z$, all $x < 0$ and all $u \in \partial j_+(z,x)$, we have $u = 0$ (see (9)) From this together with (18) and (19), we deduce that for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j_+(z,x)$, one has

$$u \leq (\theta(z) + \varepsilon) |x|^{p-1} + a_\varepsilon(z) \tag{20}$$

By hypothesis $H(j)$ (iii), for all $z \in Z \backslash D$ with $|D|_N = 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$), the function $x \to j(z,x)$ is locally Lipschitz and so, by Rademacher's theorem, it is almost everywhere differentiable. Moreover, at any such point $r \in \mathbb{R}$ of differentiability, we have

$$\frac{d}{dr} j_+(z,r) \in \partial j_+(z,r)$$

(see Clarke [18], p.32), hence

$$\frac{d}{dr} j_+(z,r) \leq (\theta(z) + \varepsilon) r^{p-1} + a_\varepsilon(z) \text{ for a.a. } z \in Z$$

(see (20)). Integrating this inequality on $[0,x]$, $x > 0$, we obtain

$$j_+(z,x) \leq \frac{1}{p}(\theta(z) + \varepsilon) x^p + a_\varepsilon(z) x \text{ for a.a. } z \in Z, \text{ all } x \geq 0 \tag{21}$$

(recall that $j_+(z,0) = 0$ a.e. on Z). So, if $x \in W_+$, we have

$$\begin{aligned}
\varphi_+(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j_+(z, x(z))\, dz \\
&\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta |x|^p\, dz - \frac{\varepsilon}{p}\|x\|_p^p - \|a_\varepsilon\|_\infty \|x\|_1 \\
&\geq \frac{\xi_0}{p}\|Dx\|_p^p - \frac{\varepsilon}{p\lambda_1}\|Dx\|_p^p - c_1\|Dx\|_p \\
&= \frac{1}{p}\left(\xi_0 - \frac{\varepsilon}{\lambda_1}\right)\|Dx\|_p^p - c_1\|Dx\|_p,
\end{aligned} \tag{22}$$

for some $c_1 > 0$ (see (49)). If we choose $0 < \varepsilon < \lambda_1 \xi_0$, from (22) and Poincaré's inequality, we infer that $\varphi_+|_{W_+}$ is coercive. Moreover, due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we verify easily that φ_+ is weakly lower semicontinuous. So by the Weierstrass theorem, we can find $x_0 \in W_+$ such that

$$-\infty < m_+ = \inf_{x \in W_+} \varphi_+(x) = \varphi_+(x_0).$$

Hypothesis $H(j)(v)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that for almost all $z \in Z$, all $0 < x \leq \delta$ and all $u \in \partial j_+(z, x)$, we have

$$(\eta_1(z) - \varepsilon) x^{p-1} \leq u \tag{23}$$

From (23) as above, we obtain

$$\frac{1}{p}(\eta_1(z) - \varepsilon) x^p \leq j_+(z, x) \text{ for a.a. } z \in Z, \text{ all } 0 < x \leq \delta. \tag{24}$$

If $u_1 \in int\ C_+$ is the L^p-normalized principal eigenfunction, we can find $\beta > 0$ small, such that

$$\beta u_1(z) \in (0, \delta] \text{ for all } z \in Z. \tag{25}$$

Then by (24) and (25) we get

$$\begin{aligned}
\varphi_+(\beta u_1) &= \frac{\beta^p}{p} \|Du_1\|_p^p - \int_Z j_+(z, \beta u_1(z))\, dz \\
&\leq \frac{\beta^p}{p} \|Du_1\|_p^p - \frac{\beta^p}{p} \int_Z (\eta_1(z) - \varepsilon) |u_1(z)|^p\, dz \\
&= \frac{\beta^p}{p} \left[\int_Z (\lambda_1 - \eta(z))\, u_1(z)^p\, dz + \varepsilon \right]
\end{aligned} \tag{26}$$

(since we assumed $\|u_1\|_p = 1$). Since $u_1(z) > 0$ for all $z \in Z$, we see that

$$\widehat{\xi} = \int_Z (\lambda_1 - \eta(z))\, u_1(z)^p\, dz < 0.$$

Thus, if we choose $\varepsilon < -\widehat{\xi}$, from (26) we see that

$$\varphi_+(\beta u_1) < 0,$$

hence

$$m_+ = \varphi_+(x_0) < 0 = \varphi_+(0), \text{ i.e. } x_0 \neq 0$$

From Clarke [?], p.52, we have

$$0 \in \partial\varphi_+(x_0) + N_{W_+}(x_0), \tag{27}$$

with $N_{W_+}(x_0)$ being the normal cone to W_+ at x_0. By definition,

$$N_{W_+}(x_0) = \left\{ u^* \in W^{-1,p'}(Z) : \langle u^*, y - x_0 \rangle \leq 0 \text{ for all } y \in W_+ \right\} \tag{28}$$

(see Gasinski-Papageorgiou [28], p.526). From (27), we can find $x^* \in \partial\varphi_+(x_0)$ such that

$$-x^* \in N_{W_+}(x_0). \tag{29}$$

From (13), we know that

$$x^* = A(x_0) - u_0 \text{ with } u_0 \in N_+(x_0).$$

So from (28) and (29), we have

$$0 \leq \langle A(x_0) - u_0, y - x_0 \rangle \text{ for all } y \in W_+. \tag{30}$$

Let $\varepsilon > 0$ and $h \in W_0^{1,p}(Z)$ be given and set

$$y = (x_0 + \varepsilon h)^+ = x_0 + \varepsilon h + (x_0 + \varepsilon h)^- \in W_+.$$

We use this as a test function in (30) and we obtain

$$0 \leq \varepsilon \langle x^*, h \rangle + \left\langle x^*, (x_0 + \varepsilon h)^- \right\rangle$$

hence

$$-\left\langle x^*, (x_0 + \varepsilon h)^- \right\rangle \leq \varepsilon \langle x^*, h \rangle. \tag{31}$$

We let $Z_\varepsilon^- = \{z \in Z : (x_0 + \varepsilon h)(z) < 0\}$. We know that

$$D\left[(x_0 + \varepsilon h)^-\right](z) = \begin{cases} -D(x_0 + \varepsilon h)(z) & \text{if } z \in Z_\varepsilon^- \\ 0 & \text{otherwise} \end{cases}. \tag{32}$$

Then

$$\begin{aligned}
&-\left\langle x^*, (x_0 + \varepsilon h)^- \right\rangle \\
&= -\left\langle A(x_0), (x_0 + \varepsilon h)^- \right\rangle + \int_Z u_0 (x_0 + \varepsilon h)^- \, dz \\
&= -\int_Z \|Dx_0\|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)^-\right)_{\mathbb{R}^N} dz + \int_Z u_0 (x_0 + \varepsilon h)^- \, dz.
\end{aligned} \tag{33}$$

We estimate both integrals in the right hand side of (33). So by using (32) we have

$$\begin{aligned}
&-\int_Z \|Dx_0\|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)^-\right)_{\mathbb{R}^N} dz \\
&= \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)\right)_{\mathbb{R}^N} dz \\
&\geq \varepsilon \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2} (Dx_0, Dh)_{\mathbb{R}^N} dz \\
&= \varepsilon \int_{Z_\varepsilon^- \cap \{x_0 > 0\}} \|Dx_0\|^{p-2} (Dx_0, Dh)_{\mathbb{R}^N} dz
\end{aligned} \tag{34}$$

since $x_0 \in W_+$ and by Stampacchia's theorem, we have

$$Dx_0(z) = 0 \text{ a.e. on } \{x_0 = 0\}$$

(see Gasinski-Papageorgiou [28], p.195). Also we have

$$\int_Z u_0 (x_0 + \varepsilon h)^- \, dz = -\int_{Z_\varepsilon^-} u_0 (x_0 + \varepsilon h) \, dz \geq 0 \tag{35}$$

(see hypothesis $H(j)(vi)$).

We return to (33) and use (34) and (35). Then

$$-\left\langle x^{*},(x_{0}+\varepsilon h)^{-}\right\rangle \geq \varepsilon\int_{Z_{\varepsilon}^{-}\cap\{x_{0}>0\}}\|Dx_{0}\|^{p-2}(Dx_{0},Dh)_{\mathbb{R}^{N}}\,dz,$$

hence

$$\langle x^{*},h\rangle \geq \int_{Z_{\varepsilon}^{-}\cap\{x_{0}>0\}}\|Dx_{0}\|^{p-2}(Dx_{0},Dh)_{\mathbb{R}^{N}}\,dz \tag{36}$$

(see (31)). Since $|Z_{\varepsilon}^{-}\cap\{x_{0}>0\}|_{N}\to 0$ as $\varepsilon\downarrow 0$, if we pass to the limit as $\varepsilon\downarrow 0$ in (36), we obtain

$$0\leq\langle x^{*},h\rangle \text{ for all } h\in W_{0}^{1,p}(Z),$$

hence

$$A(x_{0})=u_{0}. \tag{37}$$

From (37) we infer that

$$\begin{cases} -div\left(\|Dx_{0}(z)\|^{p-2}Dx_{0}(z)\right)=u_{0}(z) \text{ a.e. on } Z,\\ x_{0}|_{\partial Z}=0. \end{cases} \tag{38}$$

From (38) and nonlinear regularity theory, we have that $x_{0}\in C_{+}$, $x_{0}\neq 0$. Then (38) and hypothesis $H(j)(vi)$ imply

$$div\left(\|Dx_{0}(z)\|^{p-2}Dx_{0}(z)\right)\leq c_{0}x_{0}(z)^{p-1} \text{ a.e. on } Z. \tag{39}$$

From (39) and the nonlinear strict maximum principle of Vasquez [51], we obtain $x_{0}\in int\ C_{+}$. So from (9) we conclude that $x_{0}\in int\ C_{+}$ is a solution of problem (8).

Similarly, working with the truncated locally Lipschitz functional φ_{-} we obtain a second solution $v_{0}\in -int\ C_{+}$. ■

5 Spectrum of the negative p-Laplacian

In this section we determine some important spectral properties of the negative p-Laplacian with Dirichlet and Neumann boundary conditions. Before passing to the study of the spectral properties of the p-Laplacian let summarize the situation with the scalar eigenvaue problems (i.e., $N=1$) (for details see [27], p.93).

Consider first the case of the scalar Laplacian with Dirichlet boundary conditions

$$\begin{cases} -x''(t)=\lambda x(t) \text{ a. e. on } (0,b)\\ x(0)=x(b)=0 \end{cases} \tag{40}$$

We say thar $\lambda\in\mathbb{R}$ is an *eigenvalue* of the negative scalar Laplacian $\left(-x'',W^{1,2}(0,b)\right)$ if the problem (40) has a nontrivial solution $x\in W^{1,2}(0,b)$, which is called a corresponding *eigenfunction*.

It is wel known that $\left(-x'', W^{1,2}(0,b)\right)$ has a sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ and the eigenfunctions form an ortonormal basis of $L^2(0,b)$. Namely, the eigenvalues of $\left(-x'', W^{1,2}(0,b)\right)$ are

$$\left\{\lambda_n = \left(\frac{n\pi}{b}\right)^2 : n \geq 1\right\}$$

and the eigenfunctions are

$$\left\{u_n(t) = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi}{b} t\right)^2 : n \geq 1\right\}$$

Similarly, instead of (40) we can consider the Neumann problem

$$\begin{cases} -x''(t) = \lambda x(t) \text{ a. e. on } (0,b) \\ x'(0) = x'(b) = 0 \end{cases} \tag{41}$$

and in this case the eigenvalues are

$$\lambda_n = (\frac{n\pi}{b})^2 \text{ for } n \geq 0$$

and the eigenfunctions are

$$u_0(t) = \frac{1}{\sqrt{b}} \text{ and } u_n(t) = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi}{b} t\right)^2 \text{ for } n \geq 1$$

Finally we consider the periodic problem:

$$\begin{array}{l} -x''(t) = \lambda x(t) \text{ a. e. on } (0,b) \\ x(0) = x(b), \ x'(0) = x'(b), \end{array} \tag{42}$$

and again we can say that there exists a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ and the eigenfunctions form an ortonormal basis of $L^2(0,b)$. More precisely we have

$$\lambda_n = (\frac{n\pi}{b})^2 \text{ for } n \geq 0$$

and

$$u_0(t) = \frac{1}{\sqrt{b}} \text{ and } u_n(t) = 2\cos\left(\frac{2n\pi}{b} t\right), \text{ for } n \geq 1.$$

Now we briefly mention what is the situation with the eigenvalue problems for the scalar ordinary p-Laplacian. For details see Drabek–Manasevich [**?**] and Gasiński–Papageorgiou ([**?**], Section 6.3).

For $1 < p < \infty$ we consider the Dirichlet problem:

$$\begin{cases} -\left(|x'(t)|^{p-2} x'(t)\right)' = \lambda |x'(t)|^{p-2} x'(t) \text{ a. e. on } (0,b) \\ x(0) = x(b) = 0. \end{cases} \tag{43}$$

A number $\lambda \in \mathbb{R}$ is called an *eigenvalue* of the negative scalar p-Laplacian with Dirichlet conditions, if the problem (43) has a nontrivial solution. which is called a corresponding *eigenfunction.*

For this case, there exists eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ and the corresponding eigenfunctions u_n for $n \geq 1$, where

$$\lambda_n^D := (\frac{n\pi_p}{b})^p \text{ and } u_n^D := a \sin_p(\frac{n\pi_p t}{b}) \text{ for } n \geq 1,$$

with

$$\pi_p = 2\,(p-1)^{\frac{1}{p}} \int\limits_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi\,(p-1)^{\frac{1}{p}}}{p \sin\left(\frac{\pi}{p}\right)}$$

(observe that $\pi_2 = \pi$) and $\sin_p : \mathbb{R} \to \mathbb{R}$ is defined by

$$\int\limits_0^{\sin_p t} \frac{ds}{\left(1 - \frac{s^p}{p-1}\right)^{\frac{1}{p}}} = t, \text{ for any } t \in \left[0, \frac{\pi_p}{2}\right]$$

and then extent to all of $\mathbb{R}$ in a similar way as for $\sin(.)$ (for details we refer to Otani[43], [44] and del Pino-Elgueta-Manasevich [21]).

Next we consider the Neumann problem

$$\begin{cases} -\left(|x'(t)|^{p-2}\, x'(t)\right)' = \lambda\, |x'(t)|^{p-2}\, x'(t) \text{ a. e. on } (0,b) \\ x'(0) = x'(b) = 0. \end{cases} \tag{44}$$

A number $\lambda \in \mathbb{R}$ is called an *eigenvalue* of the negative scalar p-Laplacian with Neumann conditions, if the problem (44) has a nontrivial solution. which is called a corresponding *eigenfunction.*

For this case, there exists eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ and the corresponding eigenfunctions u_n for $n \geq 0$, where

$$\lambda_0^N = 0,\ \lambda_n^N = (\frac{n\pi_p}{b})^p \text{ and } u_0^N = c \in \mathbb{R}\backslash \{0\}, u_n^N(t) = u_n^D\left(t - \frac{b}{2n}\right), \text{ for } n \geq 1.$$

Finaly, we deal with the periodic eigenvalue problem

$$\begin{cases} -\left(|x'(t)|^{p-2}\, x'(t)\right)' = \lambda\, |x'(t)|^{p-2}\, x'(t) \text{ a. e. on } (0,b) \\ x(0) = x(b), x'(0) = x'(b). \end{cases} \tag{45}$$

Then all the eigenvalues of the ordinary p-Laplacian with periodic boundary conditions are the following:

$$\lambda_0^P = 0,\ \lambda_n^P = \lambda_{2n}^D,\ u_0^P = c \in \mathbb{R}\backslash \{0\} \text{ and } u_n^P(t) = u_{2n}^D(t), \text{ for } n \geq 1.$$

Remark 25 *The situation is much complicated if instead of the scalar ordinary p-Laplacian consideramos the vector ordinary p-Laplacian* $-\left(\|x'(t)\|^{p-2}x'(t)\right)'$ *with* $x \in W^{1,p}\left((0,b), \mathbb{R}^N\right)$. *For the vectorial ordinary p-Laplacian with periodic boundary conditions, the spectrum is far from beiing understood (see [27], p. 95).*

Now we turn our attention to the study of the spectral properties of the p-Laplacian differential operator

$$\triangle_p x = div\left(\|\nabla x\|_{\mathbb{R}^N}^{p-2} \nabla x(z)\right),\ 1 < p < \infty. \tag{46}$$

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2 boundary ∂Z, $m \in L^\infty(Z)_+$, $m \neq 0$ be a weight function. We consider two weightead (with weight m) eigenvalues problems, one with Dirichlet boundary conditions

$$\begin{cases} -\triangle_p x(z) = \lambda m(z)|x(z)|^{p-2} x(z) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0 \end{cases} \tag{47}$$

and the other with Neumann boundary conditions:

$$\begin{cases} -\triangle_p x(z) = \lambda m(z)|x(z)|^{p-2} x(z) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{cases} \tag{48}$$

Definition 26 *We say that a number* $\lambda \in \mathbb{R}$ *is an eigenvalue of (the Dirichlet p-Laplacian)* $\left(-\triangle_p, W_0^{1,p}(Z), m\right)$ *if problem* (47) *has a nontrivial solution, which is known as an eigenfunction corresponding to* λ.

Definition 27 *We say that a number* $\lambda \in \mathbb{R}$ *is an eigenvalue of (the Neumann p-Laplacian)* $\left(-\triangle_p, W^{1,p}(Z), m\right)$ *if problem* (48) *has a nontrivial solution, which is known as an eigenfunction corresponding to* λ.

Let consider first the case of Dirichlet p-Laplacian. If the weight $m = 1$ then we simply write $\left(-\triangle_p, W^{1,p}(Z)\right)$ instead of $\left(-\triangle_p, W^{1,p}(Z), 1\right)$. It is known (see [28], p. 732, 739 745) that problem (47) has a smallest eigenvalue denoted by $\lambda_1(m)$, with the following properties: $\lambda_1(m) > 0$, $\lambda_1(m)$ is isolated and simple (i.e. the corresponding eigenspace is one-dimensional). Moreover, $\lambda_1(m)$ admits the following variational characterization

$$\lambda_1(m) = \inf\left\{ \frac{\|Du\|_p^p}{\displaystyle\int_Z m|u|^p dz} : u \in W_0^{1,p}(Z),\ u \neq 0 \right\}. \tag{49}$$

In (49) the infimum is actually realized at a corresponding eigenfunction $u_1 \in C_0^1\left(\overline{Z}\right)$.

Note that if u_1 is a solution of the minimization problem (49), then so does $|u_1|$ and so we may assume that $u_1(z) \geq 0$ for all $z \in \overline{Z}$. In fact invoking the strict maximum principle of Vasquez [51], we have

$$u_1(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial u_1}{\partial n}(z) < 0 \text{ for all } z \in \partial Z. \tag{50}$$

If $m,\ m' \in L^\infty(Z)_+$, $0 \leq m(z) \leq m'(z)$ a.e. on Z with strict inequalities on sets (not necessarily the same) of positive measure, then $\lambda_1(m') < \lambda_1(m)$ (see Anane-Tsouli [7]).

If $m \equiv 1$, then we write $\lambda_1 = \lambda_1(1)$. Finally, if $u \in W_0^{1,p}(Z)$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(m)$, then $u \in C_0^1\left(\overline{Z}\right)$ must change sign. The Banach space

$$C_0^1\left(\overline{Z}\right) = \left\{x \in C^1\left(\overline{Z}\right) : x\,|_{\partial Z}= 0\right\}, \tag{51}$$

is an ordered Banach space with order cone

$$C_0^1\left(\overline{Z}\right)_+ = \left\{x \in C_0^1\left(\overline{Z}\right) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\right\}.$$

This order cone has a nonempty interior, given by

$$\begin{aligned} int\ C_0^1\left(\overline{Z}\right)_+ = \Big\{x \in C_0^1\left(\overline{Z}\right) : x(z) > 0\ \forall z \in Z \text{ and} \\ \frac{\partial x}{\partial n}(z) < 0\ \forall z \in \partial Z\Big\}. \end{aligned} \tag{52}$$

Note that from (51) and (52), we infer that $u_1 \in int\ C_0^1\left(\overline{Z}\right)_+$.

Let us also recall a few basic facts about the spectrum of the negative $p-$Laplacian with Neumann boundary condition. Details can be found in Lê [36] and Gasinski-Papageorgiou [28]. The linear subspace of $W^{1,p}(Z)$ generated by the eigenfunctions corresponding to an eigenvalue $\widehat{\lambda} \in \mathbb{R}$ is the eigenspace corresponding to $\widehat{\lambda}$ and it is denoted by $E\left(\widehat{\lambda}\right)$.

There exists the smallest eigenvalue $\widehat{\lambda}_0(m) = 0$. It is isolated and simple (i.e., $E\left(\widehat{\lambda}_0(m)\right) = \mathbb{R}$). There is a variational caracterization of $\widehat{\lambda}_0(m)$ via the Rayleigh quotient, namely

$$0 = \widehat{\lambda}_0(m) = \inf\left\{\frac{\|Dx\|_p^p}{\int_Z m\,|x|^p\,dz} : x \in W^{1,p}(Z),\ x \neq 0\right\}. \tag{53}$$

Here and in what follows, $\|.\|_p$ denotes the norm in $L^p(Z)$ or in $L^p\left(Z, \mathbb{R}^N\right)$.

Evidently, constant functions (i.e., functions in $E\left(\widehat{\lambda}_0(m)\right)$) realize the infimum. According to the Lusternik-Schnirelmann theory, in addition to $\widehat{\lambda}_0 = \widehat{\lambda}_0(m)$, we have a whole strictly increasing sequence $\left\{\widehat{\lambda}_k(m)\right\}_{k\geq 0} \subseteq \mathbb{R}_+$ of eigenvalues such that $\widehat{\lambda}_k := \widehat{\lambda}_k(m) \to \infty$ as $k \to \infty$.

These eigenvalues are called the *variational eigenvalues* or *Lusternik-Schnirelmann eigenvalues ((LS)-eigenvalues* for short) of $\left(-\triangle_p, W^{1,p}(Z), m\right)$. In what follows, by $\sigma(p,m)$ we denote the set of all eigenvalues of $\left(-\triangle_p, W^{1,p}(Z), m\right)$.

It is easy to see that $\sigma(p,m) \subseteq \mathbb{R}_+$ is closed and

$$\left\{\widehat{\lambda}_k(m)\right\}_{k\geq 0} \subseteq \sigma(p,m).$$

If $p=2$ (linear eigenvalue problem), then

$$\sigma(2,m) = \left\{\widehat{\lambda}_k(m)\right\}_{k\geq 0},$$

i.e., the variational eigenvalues are all the eigenvalues of $\left(-\triangle, H^1(Z), m\right)$.

If $p \neq 2$ (nonlinear eigenvalue problem), then we do not know if this is the case. However, since $\sigma(p,m)$ is closed, we can define

$$\widehat{\lambda}_1^*(m) = \inf\left\{\lambda : \lambda \in \sigma(p,m), \lambda > \widehat{\lambda}_0\right\}$$

and this is the second eigenvalue of $\left(-\triangle_p, W^{1,p}(Z), m\right)$. We have

$$\widehat{\lambda}_1^*(m) = \widehat{\lambda}_1(m).$$

So, the first two eigenvalues of $\left(-\triangle_p, W^{1,p}(Z), m\right)$ coincide with the first two variational eigenvalues.

In the remaing of this section we present some further results concerning the spectrum of $\left(-\triangle_p, W^{1,p}(Z), m\right)$, obtained in the joint paper [4], which will be used in the study of boundary value problems.from the next sections and are also of independent interest.

In what follows, if $m \equiv 1$, then we write $\lambda_1(p) = \widehat{\lambda}_1(1)$ to emphasize the dependence on p, since we will investigate the dependence of the eigenvalue on $p \in (1, +\infty)$. Let

$$\varphi_m(x) = \int_Z m|x|^p\,dz,\ \ \psi_m(x) = \int_Z m|x|^p\,dz + \|Dx\|_p^p \text{ for all } x \in W^{1,p}(Z),$$

$$S(\psi_m) = \left\{x \in W^{1,p}(Z) : \psi_m(x) = 1\right\} \tag{54}$$

and

$$\mathcal{A}_k = \{C \subseteq S(\psi_m) :\ C \text{ is compact, symmetric and } \widetilde{\gamma}(C) \geq k\}$$

where $\widetilde{\gamma}$ is the Krasnoselskii genus [28] . From the Lusternik-Schnirelmann theory we have the following characterization of the variational eigenvalues $\left\{\widehat{\lambda}_k(m)\right\}_{k\geq 0}$ (for this reason they are called *variational*):

$$\frac{1}{\widehat{\lambda}_k(m)+1} = \sup_{C\in\mathcal{A}_k} \inf_{x\in C} \varphi_m(x) \text{ for all } k \geq 1. \tag{55}$$

Note that these sup inf - expresions are nonlinear versions of the well-known minimax characterizations of the eigenvalues of $\left(-\triangle, H^1(Z)\right)$ (linear eigenvalue problem) due to Courant (see for example Gasinski-Papageorgiou [28], p.718).

Of all of the eigenfunctions of $\left(-\triangle_p, W^{1,p}(Z), m\right)$, only the principal ones (i.e., the ones belonging to the eigenspace $E\left(\widehat{\lambda}_0(m)\right) = E(0)$) do not change sign. All the others are nodal functions. In our study we will work with the Sobolev space $W^{1,p}(Z)$, whose norm is given by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}}.$$

The variational characterization of $\lambda_1(p)$ given in (53) is not convenient for our purposes, and for this reason we need to come up with new ones. The first alternative variational characterization of $\lambda_1(p)$ is well known (see for example Gasinski-Papageorgiou [28]).

Proposition 28 *If for* $1 < p < \infty$,

$$C_1(p) := \left\{ x \in W^{1,p}(Z) : \|x\|_p = 1, \int_Z |x(z)|^{p-2} x(z)\, dz = 0 \right\},$$

then

$$\lambda_1(p) = \min\left\{\|Du\|_p^p : u \in C_1(p)\right\}.$$

Let

$$u_0(z) = \frac{1}{|Z|_N^{\frac{1}{p}}} \text{ for all } z \in \overline{Z}$$

be the normalized principal eigenfunction of $\left(-\triangle_p, W^{1,p}(Z), m\right)$, where by $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Inspired by a corresponding result for the Dirichlet $p-$Laplacian (see Cuesta-de Figueiredo-Gossez [?]), we can prove a third variational characterization of $\lambda_1(p)$.

In what follows

$$\partial B_1^{L^p} = \left\{x \in L^p(Z) : \|x\|_p = 1\right\}.$$

Proposition 29 *If* $S^p = W^{1,p}(Z) \cap \partial B_1^{L^p}$ *and*

$$\Gamma_p = \left\{\gamma \in C([-1,1], S^p) : \gamma(-1) = -u_0, \gamma(1) = u_0\right\},$$

then

$$\lambda_1(p) = \min_{\gamma \in \Gamma_p} \max_{-1 \le t \le 1} \|D\gamma(t)\|_p^p.$$

In what follows, we set

$$S_c^p := S^p \cap C^1\left(\overline{Z}\right)$$

furnished with the $C^1\left(\overline{Z}\right)$ topology. Recall that

$$S^p = W^{1,p}(Z) \cap \partial B_1^{L^p}$$

furnished with the $W^{1,p}(Z)$-norm. Evidently, S_c^p is dense in S^p for the $W^{1,p}(Z)$-norm. Therefore $C([-1,1], S_c^p)$ is dense in $C([-1,1], S^p)$.

Next, we consider the map $p \to \lambda_1(p)$. Using the last two propositions, we prove that this map is continuous.

Proposition 30 *The map $\lambda_1 : (1,\infty) \to (0,\infty)$ is continuous.*

From Section 2, we know that for every $1 < p < \infty$, the principal eigenvalue $\lambda_0(p) = 0$ is isolated. We will show that this isolation of the principal eigenvalue, is uniform with respect to p belonging in a bounded closed subinterval of $(1,p)$.

Proposition 31 *If $1 < a < b < \infty$ and $I = [a,b]$, then one can find $\delta > 0$ such that for all $p \in I$, $\lambda_1(p) \notin (0,\delta)$.*

Proof. Suppose we can find a sequence $\{p_n\}_{n\geq 1} \subseteq I$ such that

$$\lambda_1(p_n) \to 0 \text{ as } n \to \infty.$$

Since I is compact, we may assume that $p_n \to p \in I$. By virtue of Proposition 3, we have that $\lambda_1(p) = 0$. Since the principal eigenvalue $\lambda_0(p) = 0$ is isolated, we have a contradiction. ■

The next proposition establishes a monotonicity property of $\widehat{\lambda}_1(m)$ with respect to the weight function $m \in L^\infty(Z)_+$. It extends a corresponding result proved for the Dirichlet eigenvalues by Anane and Tsouli [7], and its proof can be found in [1].

Proposition 32 *If $m,\ m_1 \in L^\infty(Z)_+$, $m \neq 0$ and $m(z) < m_1(z)$ a.e. on Z, then*

$$\widehat{\lambda}_1(m_1) < \widehat{\lambda}_1(m).$$

6 Method of upper-lower solutions

The method of upper and lower solutions provides an effective tool to produce existence theorems for first and second order initial and boundary value problems and to generate monotone iterative techniques which provide constructive methods (amenable to numerical treatment), to obtain solutions. We shall apply this method to several boundary value problems. One is this:

$$\left\{ \begin{array}{l} -\triangle_p x(z) + \beta |x(z)|^{p-2} x(z) = f(z, x(z)) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right. \tag{56}$$

where $Z \subseteq \mathbb{R}^n$ is a bounded domain with a C^2 boundary ∂Z, $\triangle_p x(z) = div\left(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)\right)$, $1 < p < \infty$, is the $p-$Laplacian differential operator, $\beta > 0$ and $f(z,x)$ is a Carathéodory nonlinearity. We want to prove a *three solutions theorem* for problem (56), when the nonlinearity $f(z,.)$ exhibits a $(p-1)$ -sublinear behavior near the origin (concave nonlinearity).

Recently, there have been some multiplicity results for Neumann problems driven by the $p-$Laplacian differential operator. We mention the works of Anello [8], Binding-Drabek-Huang [10], Bonanno-Candito [11], Faraci [24], Filippakis-Gasinski-Papageorgiou [25], Motreanu-Papageorgiou [11], Ricceri [48] and Wu-Tan [54]. In Anello [8], Bonanno-Candito [11], Faraci [24] and Ricceri [49], the authors consider nonlinear eigenvalue problems and prove the existence of multiple solutions when the nonlinearity is oscillating and the parameter belongs to an open interval in $\mathbb{R}_+$. In these works, the key assumption is that $p > N$ (low dimensional problem), which implies that the Sobolev space $W^{1,p}(Z)$ is embedded compactly in $C\left(\overline{Z}\right)$. The approach in all these papers is essentially similar, and is based on an abstract variational principle due to Ricceri [48]. In Wu-Tan [54], it is again assumed that $p > N$ and the approach (which is variational) is based on the critical point theory. Binding-Drabek-Huang [10] considered problems with a particular right-hand side nonlinearity, of the form $\lambda a(z)|x|^{p-2}x + b(z)|x|^{q-2}x$, with $a,\ b \in L^\infty(Z)$, $\lambda \in \mathbb{R}$, $1 < p < N$ and $1 < q < p^*$, where p^* is the critical Sobolev exponent given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases} \tag{57}$$

They prove the existence of one or two positive solutions. Finally, we should also mention the recent work [3], which is concerned with problem (56) with a $p-$ superlinear potential $F(z,x) = \int\limits_0^x f(s,x)\,ds$ (where $f(z,.)$ satisfies the Ambrosetti-Rabinowitz condition). The authors prove multiplicity theorems, providing precise information about the sign of the solutions.

None of the aforementioned works treats nonlinearities which are concave near the origin. Problems with concave nonlinearities were considered in the context of semilinear problems (i.e., $p = 2$) or Dirichlet problems, by de Paiva-Massa [20], Li-Wu-Zhou [37], Perera [46] and Wu-Yang [53]. For Dirichlet problems with the p-Laplacian, we mention the work of Garcia Azorero-Manfredi-Peral Alonso [26], where a nonlinear eigenvalue problem is considered, with a nonlinearity of the form $\lambda|x|^{r-2}x + |x|^{q-2}x$, with $\lambda > 0$ and $1 < r < p < q < p^*$ (concave-convex nonlinearity). Their work extended earlier results for the semilinear case by Ambrosetti-Brezis-Cerami [6].

Our approach here is different from all of the above works. It combines variational techniques with the method of upper-lower solutions and with Morse theory (in particular, critical groups).

In this section we produce two nontrivial smooth solutions of constant sign (one positive and the other negative) by employing variational arguments in combination with the method of upper-lower solutions. The third solution will be obtained in the last section of this paper by using suitable tools from Morse theory.

In the analysis of problem (56) we will use the following two spaces:

$$W_n^{1,p}(Z)=\left\{x\in W^{1,p}(Z):x_k\to x \text{ in } W^{1,p}(Z),\ x_k\in C^\infty\left(\overline{Z}\right),\frac{\partial x_k}{\partial n}=0 \text{ on } \partial Z\right\}$$

and

$$C_n^1\left(\overline{Z}\right)=\left\{x\in C^1\left(\overline{Z}\right):\frac{\partial x}{\partial n}=0 \text{ on } \partial Z\right\},$$

where by $\overline{Z}$ we denote the closure of the domain Z. Both are ordered Banach spaces, with order cones given by

$$W_+=\left\{x\in W_n^{1,p}(Z):x(z)\geq 0 \text{ a.e. on } Z\right\}$$

and respectively

$$C_+=\left\{x\in C_n^1\left(\overline{Z}\right):x(z)\geq 0 \text{ for all } z\in\overline{Z}\right\}.$$

We know that $int\ C_+\neq\varnothing$ (where int stands for the interior), with

$$intC_+=\left\{x\in C_+:x(z)>0 \text{ for all } z\in\overline{Z}\right\}.$$

In what follows, by $\|.\|_p$ we denote the norm of $L^p(Z)$ (or $L^p\left(Z,\mathbb{R}^N\right)$), and by $\|.\|$ the norm of $W^{1,p}(Z)$. The norm of $W_n^{1,p}(Z)$ is also denoted by $\|.\|$.

The next result, due to Aizicovici, Papageorgiou and Staicu [3], compares $C_n^1\left(\overline{Z}\right)$ and $W_n^{1,p}(Z)-$ local minimizers for a large class of energy functionals. It extends to Neumann problems earlier results by Brezis-Nirenberg [13] (for $p=2$) and by Garcia Azorero-Manfredi-Peral Alonso [26] (for $p\neq 2$), which were concerned with Dirichlet boundary conditions.

So, consider a nonlinearity $\widehat{f}:Z\times\mathbb{R}\to\mathbb{R}$ satisfying the following hypotheses:

(H_0) (i) for all $x\in\mathbb{R}$, $z\to\widehat{f}(z,x)$ is measurable;

(ii) for almost all $z\in Z$, $x\to\widehat{f}(z,x)$ is continuous;

(iii) for almost all $z\in Z$ and all $x\in\mathbb{R}$

$$\left|\widehat{f}(z,x)\right|\leq\widehat{a}(z)+\widehat{c}|x|^{r-1},$$

where $\widehat{a}\in L^\infty(Z)_+$, $\widehat{c}>0$ and $1<r<p^*$, with p^* defined by (**??**).

Let $\widehat{F}(z,x)=\int_0^x\widehat{f}(z,s)\,ds$ and consider the functional $\widehat{\varphi}:W_n^{1,p}(Z)\to\mathbb{R}$ defined by

$$\widehat{\varphi}(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z\widehat{F}(z,x(z))\,dz \text{ for all } x\in W_n^{1,p}(Z).$$

Evidently $\widehat{\varphi}\in C^1\left(W_n^{1,p}(Z)\right)$.

Proposition 33 *Let (H_0) be satisfied. If $x_0 \in W_n^{1,p}(Z)$ is a local $C_n^1\left(\overline{Z}\right)-$ minimizer of $\widehat{\varphi}$, i.e., there exists $\rho_1 > 0$ such that*

$$\widehat{\varphi}(x_0) \le \widehat{\varphi}(x_0 + h) \text{ for all } h \in C_n^1\left(\overline{Z}\right), \ \|h\|_{C_n^1\left(\overline{Z}\right)} \le \rho_1,$$

then $x_0 \in C_n^1\left(\overline{Z}\right)$ and it is a local $W_n^{1,p}(Z)-$ minimizer of $\widehat{\varphi}$, i.e., there exists $\rho_2 > 0$ such that

$$\widehat{\varphi}(x_0) \le \widehat{\varphi}(x_0 + h) \text{ for all } h \in W_n^{1,p}(Z), \ \|h\| \le \rho_2.$$

Next let us recall the notions of upper and lower solutions for problem (56).

Definition 34 (a) *An* upper solution *for problem* (56) *is a function $\overline{x} \in C^1\left(\overline{Z}\right)$ such that*

$$\frac{\partial \overline{x}}{\partial n} \ge 0 \text{ on } \partial Z$$

and

$$\int_Z \|D\overline{x}\|_{\mathbb{R}^N}^{p-2} (D\overline{x}, Dh)_{\mathbb{R}^N} dz + \beta \int_Z |\overline{x}|^{p-2} \overline{x} h dz \ge \int_Z f(z, \overline{x}) h dz$$

for all $h \in W_+$. We say that $\overline{x}$ is a strict upper solution *for problem* (**??**), *if it is an upper solution but it is not a solution of* (56).

(b) *A* lower solution *for problem* (56) *is a function $\underline{x} \in C^1\left(\overline{Z}\right)$ such that*

$$\frac{\partial \underline{x}}{\partial n} \le 0 \text{ on } \partial Z$$

and

$$\int_Z \|D\underline{x}\|_{\mathbb{R}^N}^{p-2} (D\underline{x}, Dh)_{\mathbb{R}^N} dz + \beta \int_Z |\underline{x}|^{p-2} \underline{x} h dz \le \int_Z f(z, \underline{x}) h dz$$

for all $h \in W_+$. We say that $\underline{x}$ is a strict lower solution*, if it is a lower solution but it is not a solution of* (56).

Assume $1 < p < \infty$ and the hypotheses on the nonlinearity $f(z,x)$ are the following:

$\mathbf{H}(f)$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and

- (i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;
- (ii) for almost all $z \in Z$, $x \to f(z,x)$ is continuous;
- (iii) for every $\rho > 0$, there exists $a_\rho \in L^\infty(Z)_+$ such that

$$|f(z,x)| \le a_\rho(z) \text{ for a.a. } z \in Z \text{ and all } |x| \le \rho.$$

- (iv) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \le \beta$ a.e. on Z, with strict inequality on a set of positive measure, and if $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\limsup_{|x| \to \infty} \frac{pF(z,x)}{|x|^p} \le \theta(z) \text{ uniformly for a.a. } z \in Z;$$

(v) there exist $\delta > 0$, $r \in (1,p)$ and $c_0 > 0$ such that

$$c_0 |x|^r \leq F(z,x) \text{ for a.a. } z \in Z \text{ and all } |x| \leq \delta;$$

(vi) for almost all $z \in Z$, we have

$$f(z,x)\, x \geq 0 \text{ for all } x \in \mathbb{R} \qquad \text{(sign condition)}$$

and

$$pF(z,x) - f(z,x)\, x > 0 \text{ for all } x \neq 0.$$

Remark 35 *Hypothesis* $\mathbf{H}(f)(v)$ *implies that the nonlinearity* $f(z,.)$ *exhibits an* $(r-1)-$ *sublinear growth near the origin (concave nonlinearity near the origin). For example, the nonlinearity*

$$f(x,x) = \theta(z) |x|^{p-2} x + |x|^{r-2} x$$

with $1 < r < p$ *and* $\theta \in L^\infty(Z)_+$ *as in assumption* $\mathbf{H}(f)(iv)$, *satisfies hypotheses* $\mathbf{H}(f)$.

First, we will produce a strict upper solution of (56). By virtue of hypotheses $\mathbf{H}(f)(iii)$, (iv) and (vi), given $\varepsilon > 0$, we can find $\xi_\varepsilon \in L^\infty(Z)_+$, $\xi_\varepsilon \neq 0$ and $\eta_\varepsilon > 0$ such that

$$(\theta(z) + \varepsilon)\, x^{p-1} + \xi_\varepsilon(z) - f(z,x) \geq \eta_\varepsilon > 0 \text{ for a.a. } z \in Z \text{ and all } x \geq 0. \quad (58)$$

To produce a strict upper solution for problem (56), we will need the following lemma, which underlines the significance of the nonuniform resonance hypothesis $\mathbf{H}(f)(iv)$.

Lemma 36 *If* $\theta \in L^\infty(Z)_+$, $\theta(z) \leq \beta$ *a.e. on* Z, *with strict inequality on a set of positive measure, then there exists* $\widehat{\xi}_0 > 0$ *such that*

$$\psi(x) = \|Dx\|_p^p + \beta \|x\|_p^p - \int_Z \theta(z) |x(z)|^p \, dz \geq \widehat{\xi}_0 \|x\|^p \text{ for all } x \in W^{1,p}(Z).$$

Proof. Note that $\psi \geq 0$. We argue by contradiction. So, suppose that the lemma is not true. Exploiting the p-homogeneity of ψ, we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(Z)$ such that

$$\|x_n\| = 1 \text{ and } \psi(x_n) \downarrow 0.$$

By passing to a suitable subsequence we may assume that

$$x_n \xrightarrow{w} x \text{ in } W^{1,p}(Z) \text{ and } x_n \to x \text{ in } L^p(Z).$$

Then we have

$$\|Dx\|_p^p \leq \liminf_{n\to\infty} \|Dx_n\|_p^p, \; \beta \|x_n\|_p^p \to \beta \|x\|_p^p$$

and

$$\int_Z \theta(z)|x_n(z)|^p\,dz \to \int_Z \theta(z)|x(z)|^p\,dz.$$

So, in the limit as $n \to \infty$, we obtain

$$\|Dx\|_p^p + \beta\|x\|_p^p \le \int_Z \theta(z)|x(z)|^p\,dz,$$

hence

$$\|Dx\|_p^p \le \int_Z (\theta(z) - \beta)|x(z)|^p\,dz \le 0, \tag{59}$$

therefore

$$x \equiv c \in \mathbb{R}.$$

If $c = 0$, then $\|Dx_n\|_p \to 0$ and so $x_n \to 0$ in $W^{1,p}(Z)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \ge 1$. So, $c \ne 0$. From (59), we have

$$0 \le |c|^p \int_Z (\theta(z) - \beta)\,dz < 0,$$

again a contradiction. This proves the Lemma. ■

Proposition 37 *If hypotheses* $\mathbf{H}(f)$ *hold, then problem* (56) *admits a strict upper solution* $\overline{x} \in$ *int* C_+.

Proof. Consider the nonlinear operator $\widehat{K}_p : L^p(Z) \to L^{p'}(Z)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ defined by

$$\widehat{K}_p(x)(.) = |x(.)|^{p-2}x(.) \text{ for all } x \in L^p(Z).$$

Clearly $\widehat{K}_p$ is continuous and bounded (i.e., it maps bounded sets to bounded ones). Moreover, by virtue of the compact embedding of $W^{1,p}(Z)$ into $L^p(Z)$, it follows that

$$K_p = \widehat{K}_p\,|_{W^{1,p}(Z)} \colon W^{1,p}(Z) \to W^{1,p}(Z)^*$$

is completely continuous (i.e., it is sequentially weakly-strongly continuous). Therefore, by the previous Remark, the map $V : W^{1,p}(Z) \to W^{1,p}(Z)^*$ defined by

$$V(x) = A(x) + \beta K_p(x) - (\theta(.) + \varepsilon)K_p(x)$$

is pseudomonotone. Also, for every $x \in W^{1,p}(Z)$, we have

$$\langle V(x), x\rangle = \|Dx\|_p^p + (\beta - \varepsilon)\|x\|_p^p - \int_Z \theta(z)|x(z)|^p\,dz \ge \left(\widehat{\xi}_0 - \varepsilon\right)\|x\|^p \tag{60}$$

(see Lemma 48). Choosing $0 < \varepsilon < \widehat{\xi}_0$, from(**??**) we infer that V is coercive. But a pseudomonotone coercive operator is surjective (see Gasinski-Papageorgiou [28], p.336). Therefore, we can find $\overline{x} \in W^{1,p}(Z)$ such that

$$V(\overline{x}) = A(\overline{x}) + \beta K_p(\overline{x}) - (\theta + \varepsilon)K_p(\overline{x}) = \xi_\varepsilon, \tag{61}$$

where ξ_ε is as in (58). Since $\xi_\varepsilon \neq 0$, (61) implies that $\overline{x} \neq 0$. Recall that

$$\overline{x} = \overline{x}^+ - \overline{x}^-, \text{ with } \overline{x}^+ = \max\{\overline{x}, 0\} \text{ and } \overline{x}^- = -\min\{\overline{x}, 0\}.$$

On (61) we act with the test function $-\overline{x}^- \in W_n^{1,p}(Z)$ and we obtain

$$\left\|D\overline{x}^-\right\|_p^p + \beta\left\|\overline{x}^-\right\|_p^p - \int_Z \theta(z)\left|\overline{x}^-(z)\right|^p dz - \varepsilon\left\|\overline{x}^-\right\|^p \leq 0,$$

hence

$$\left(\widehat{\xi}_0 - \varepsilon\right)\left\|\overline{x}^-\right\|^p \leq 0 \tag{62}$$

(see Lemma 48). Inasmuch as $\varepsilon < \widehat{\xi}_0$, from (62) it follows that $\overline{x}^- = 0$, hence $\overline{x} \geq 0$, $\overline{x} \neq 0$. On account of (61) and the nonlinear Green identity (cf. Motreanu-Papageorgiou [41]), we get

$$\begin{cases} -\triangle_p \overline{x}(z) + \beta \overline{x}(z)^{p-1} = (\theta(z) + \varepsilon)\overline{x}(z)^{p-1} + \xi_\varepsilon(z) \text{ a.e. on } Z, \\ \frac{\partial \overline{x}}{\partial n} = 0 \text{ on } \partial Z. \end{cases} \tag{63}$$

From (63) and Theorem 7.1, p.286 of Ladyzhenskaya-Uraltseva [35], we deduce that $\overline{x} \in L^\infty(Z)$. Then, invoking Theorem 2 of Lieberman [37], we infer that $\overline{x} \in C_+$.

Note that (63) implies

$$\triangle_p \overline{x}(z) \leq \beta \overline{x}(z)^{p-1} \text{ a.e. on } Z.$$

Hence, by virtue of the nonlinear strong maximum principle of Vazquez [51], we obtain $\overline{x}(z) > 0$ for all $z \in Z$. Suppose that for some $z_0 \in \partial Z$, we have $\overline{x}(z_0) = 0$. Then, from Vazquez [51] (Theorem 5), it follows that

$$\frac{\partial \overline{x}}{\partial n}(z_0) < 0,$$

which contradicts (63). This proves that $\overline{x}(z) > 0$ for all $z \in \overline{Z}$, i.e., $\overline{x} \in int\ C_+$. Because of (58), we see that $\overline{x} \in int\ C_+$ is a strict upper solution for problem (56) in the sense of Definition 1(a). ∎

Let $g \in L^\infty(Z)$ and consider the following Neumann problem

$$\begin{cases} -\triangle_p x(z) + \beta|x(z)|^{p-2} x(z) = g(z) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{cases} \tag{64}$$

From the maximal monotonicity and coercivity of the operator $x \to A(x) + \beta K_p(x)$, we infer that the problem (64) has a solution $S(g) \in W_n^{1,p}(Z)$, which is unique due to the strict monotonicity of the operator. Moreover, the nonlinear regularity theory implies that $S(g) \in C_n^1\left(\overline{Z}\right)$. We examine the monotonicity properties of the map $g \to S(g)$.

Lemma 38 *The map $S : L^\infty(Z) \to C^1\left(\overline{Z}\right)$ is increasing, i.e., if $g_1 \leq g_2$ in $L^\infty(Z)$, then $S(g_1) \leq S(g_2)$ in $C^1\left(\overline{Z}\right)$.*

Proof. Suppose that $g_1, g_2 \in L^\infty(Z)$ and assume that $g_1 \le g_2$ in $L^\infty(Z)$ (i.e., $g_1(z) \le g_2(z)$ a.e. on Z). Set $x_1 = S(g_1)$, $x_2 = S(g_2)$. Then

$$A(x_1) + \beta K_p(x_1) = g_1 \text{ and } A(x_2) + \beta K_p(x_2) = g_2.$$

We have

$$\begin{aligned}&\left\langle A(x_2) - A(x_1), (x_1 - x_2)^+ \right\rangle + \beta \textstyle\int_{\{x_1 > x_2\}} \left(|x_2|^{p-2} x_2 - |x_1|^{p-2} x_1 \right)(x_1 - x_2)\, dz \\ &= \textstyle\int_Z (g_2 - g_1)(x_1 - x_2)^+ \, dz \ge 0\end{aligned}$$

hence

$$\begin{aligned}&\textstyle\int_{\{x_1 > x_2\}} \left(\|Dx_2\|_{\mathbb{R}^N}^{p-2} Dx_2 - \|Dx_1\|_{\mathbb{R}^N}^{p-2} Dx_1, Dx_1 - Dx_2 \right)_{\mathbb{R}^n} dz \\ &+\beta \textstyle\int_{\{x_1 > x_2\}} \left(|x_2|^{p-2} x_2 - |x_1|^{p-2} x_1 \right)(x_1 - x_2)\, dz \ge 0.\end{aligned} \tag{65}$$

But, due to the strict monotonicity of the map $\mathbb{R}^N \ni \xi \to \|\xi\|_{\mathbb{R}^N}^{p-2} \xi$ and $\mathbb{R} \ni y \to |y|^{p-2} y$, the left hand side of (65) is strictly negative, a contradiction unless

$$|\{x_1 > x_2\}|_N = 0,$$

where by $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Hence $x_1 \le x_2$. ■

Note that $x \equiv 0$ is a solution of the problem (56). We truncate the nonlinearity $f(z,x)$ at the pair $\{0, \overline{x}\}$, namely we introduce

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x \le 0 \\ f(z,x) & \text{if } 0 \le x \le \overline{x}(z) \\ f(z, \overline{x}(z)) & \text{if } \overline{x}(z) \le x. \end{cases}$$

Evidently, $\widehat{f}_+(z,x)$ is a Carathédory function, i.e., for all $x \in \mathbb{R}$, $z \to \widehat{f}_+(z,x)$ is measurable and for almost all $z \in Z$, $x \to \widehat{f}_+(z,x)$ is continuous. We set

$$\widehat{F}_+(z,x) = \int_0^x \widehat{f}_+(z,s)\, ds$$

(the primitive of $\widehat{f}_+(z,.)$) and consider the functional $\widehat{\varphi}_+ : W_n^{1,p}(Z) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_+(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\beta}{p}\|x\|_p^p - \int_Z \widehat{F}_+(z, x(z))\, dz \text{ for all } x \in W_n^{1,p}(Z).$$

We also consider $\varphi : W_n^{1,p}(Z) \to \mathbb{R}$, the Euler functional for the problem (**??**), defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\beta}{p}\|x\|_p^p - \int_Z F(z, x(z))\, dz \text{ for all } x \in W_n^{1,p}(Z).$$

Clearly, $\widehat{\varphi}_+, \varphi \in C^1\left(W_n^{1,p}(Z)\right)$.

Proposition 39 *If hypotheses* $\mathbf{H}(f)$ *hold, then problem* (56) *admits a solution* $x_0 \in int\ C_+$*, which is a local minimizer of* φ.

Proof. Exploiting the compact embedding of $W_n^{1,p}(Z)$ into $L^p(Z)$, we can easily check that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. Moreover, note that we can find $M_1 > 0$ such that

$$\left| \int_Z \widehat{F}_+ (z, x(z))\, dz \right| \leq M_1 \text{ for all } x \in W_n^{1,p}(Z).$$

Hence $\widehat{\varphi}_+$ is coercive. Invoking the theorem of Weierstrass (see [28], p.711), we can find $x_0 \in W_n^{1,p}(Z)$ such that

$$\widehat{\varphi}_+(x_0) = \inf \left\{ \widehat{\varphi}_+(x) : x \in W_n^{1,p}(Z) \right\}. \tag{66}$$

We claim that $x_0 \neq 0$. To this end, let $\delta > 0$ be as in hypothesis $\mathbf{H}(f)(v)$ and let $c \in (0, \delta]$. Then

$$\begin{aligned} \widehat{\varphi}_+(c) &= \tfrac{\beta}{p} c^p |Z|_N - \int_Z F(z,c)\, dz \\ &\leq \tfrac{\beta}{p} c^p |Z|_N - c_0 c^r |Z|_N \text{ (see hypothesis } \mathbf{H}(f)(v)) \\ &= |Z|_N\, c^r \left(\tfrac{\beta}{p} c^{p-r} - c_0 \right) \end{aligned} \tag{67}$$

Since $p > r$, if we choose $c \in (0, \delta]$ small, then from (66) and (67) it follows that

$$\widehat{\varphi}_+(x_0) \leq \widehat{\varphi}_+(c) < 0$$

hence

$$x_0 \neq 0. \tag{68}$$

From (66), we have

$$\widehat{\varphi}'_+(x_0) = 0,$$

hence

$$A(x_0) + \beta K_p(x_0) = \widehat{N}_+(x_0), \tag{69}$$

where $\widehat{N}_+(x)(.) := \widehat{f}_+(., x(.))$ for all $x \in W_n^{1,p}(Z)$. On (69), we act with the test function $-x_0^- \in W_n^{1,p}(Z)$ and obtain

$$\gamma_0 \left\| x_0^- \right\| \leq 0 \text{ with } \gamma_0 = \min\{\beta, 1\}$$

hence

$$x_0^- = 0, \text{ i.e., } x_0 \geq 0,\ x_0 \neq 0 \text{ (see (68))}.$$

From (69) it follows that

$$\left\{ \begin{array}{l} -\triangle_p x_0(z) + \beta x_0(z)^{p-1} = \widehat{f}_+(z, x_0(z)) \text{ a.e. on } Z, \\ \frac{\partial x_0}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right. \tag{70}$$

The nonlinear regularity theory implies that $x_0 \in C_+$. Due to the sign condition (see hypothesis $\mathbf{H}(f)(vi)$), we have

$$\widehat{f}_+(z, x_0(z)) \geq 0 \text{ a.e. on } Z. \tag{71}$$

From (70) and (71) it follows that

$$\triangle_p x_0(z) \leq \beta x_0(z)^{p-1} \text{ a.e. on } Z,$$

which, by virtue of the nonlinear maximum principle of Vasquez [?], implies that

$$x_0 \in int\ C_+.$$

From Proposition 4 we know that $\overline{x} \in int\ C_+$ is a strict upper solution for problem (56). So, according to Definition 1(a), we have

$$A(\overline{x}) + \beta K_p(\overline{x}) > N(\overline{x}) = \widehat{N}_+(x_0) \text{ in } W_n^{1,p}(Z)^*, \tag{72}$$

where $N(x)(.) = f(., x(.))$ for all $x \in W_n^{1,p}(Z)$. From (69) and (72) we obtain

$$A(\overline{x}) - A(x_0) + \beta(K_p(\overline{x}) - K_p(x_0)) > \widehat{N}_+(\overline{x}) - \widehat{N}_+(x_0) \text{ in } W_n^{1,p}(Z)^*. \tag{73}$$

On (73), we act with the test function $(x_0 - \overline{x})^+ \in W_n^{1,p}(Z)$. Then, arguing as in the proof of Lemma 48, we infer that

$$|\{x_0 > \overline{x}\}|_N = 0,$$

therefore

$$x_0 \leq \overline{x}.$$

Hence (70) becomes

$$\begin{cases} -\triangle_p x_0(z) + \beta x_0(z)^{p-1} = f(z, x_0(z)) \text{ a.e. on } Z, \\ \frac{\partial x_0}{\partial n} = 0 \text{ on } \partial Z. \end{cases}$$

Let $0 < \delta < \min_{\overline{Z}} \overline{x}$ and consider $v_\delta = \overline{x} - \delta \in int\ C_+$. Then

$$-\triangle_p v_\delta(z) + \beta v_\delta(z)^{p-1} \geq -\triangle_p \overline{x}(z) + \beta \overline{x}(z)^{p-1} - \sigma(\delta), \tag{74}$$

with $\sigma \in C(\mathbb{R}_+)$, $\sigma \geq 0$ and $\sigma(\delta) \to 0^+$ as $\delta \to 0^+$. Choosing $\delta > 0$ small and using (58), we have

$$(\theta(z) + \varepsilon)\overline{x}(z)^{p-1} + \xi_\varepsilon(z) - \sigma(\delta) \geq f(z, x_0(z)) + \frac{\eta_\varepsilon}{2} \text{ for a.a. } z \in Z. \tag{75}$$

From (63), (74) and (75), it follows that for $\delta > 0$ small, we have

$$\begin{aligned} h_\delta(z) &= -\triangle_p v_\delta(z) + \beta v_\delta(z)^{p-1} \\ &> f(z, x_0(z)) = -\triangle_p x_0(z) + \beta x_0(z)^{p-1} \text{ a.e. on } Z. \end{aligned} \tag{76}$$

Since $h_\delta,\ f\left(.,x_0\left(.\right)\right) \in L^\infty\left(Z\right)$, from (76) and Lemma 48 we infer that for $\delta > 0$ small

$$x_0\left(z\right) \leq v_\delta\left(z\right) \text{ for all } z \in \overline{Z},$$

hence

$$\overline{x}\left(z\right) - x_0\left(z\right) \geq \delta > 0 \text{ for all } z \in \overline{Z},$$

therefore

$$\overline{x} - x_0 \in int\ C_+.$$

Inasmuch as $x_0 \in int\ C_+$, we can find $r > 0$ small such that

$$\widehat{\varphi}_+ \mid_{\overline{B}_r^{C_0^1\left(\overline{Z}\right)}\left(x_0\right)} = \varphi \mid_{\overline{B}_r^{C_0^1\left(\overline{Z}\right)}\left(x_0\right)},$$

hence $x_0 \in int\ C_+$ is a local $C_n^1\left(\overline{Z}\right)$ $-$minimizer of φ. Invoking Proposition 45, we conclude that $x_0 \in int\ C_+$ is a local $W_n^{1,p}\left(Z\right)$ $-$minimizer of φ, and of course it solves problem $\left(56\right)$. ■

We repeat the same process on the negative half-axis. So, because of hypotheses $\mathbf{H}\left(f\right)\left(iii\right)$, $\left(iv\right)$ and $\left(vi\right)$, given $\varepsilon > 0$, we can find $\gamma_\varepsilon \in L^\infty\left(Z\right)_+$, $\gamma_\varepsilon \neq 0$ and $\widehat{\eta}_\varepsilon > 0$ such that

$$\left(\theta\left(z\right) + \varepsilon\right)\left|x\right|^{p-2} x - \gamma_\varepsilon\left(z\right) \leq f\left(z,x\right) - \widehat{\eta}_\varepsilon \text{ for a.a. } z \in Z \text{ and all } x \leq 0. \quad (77)$$

We consider the following auxiliary Neumann problem

$$\left\{ \begin{array}{l} -\triangle_p v\left(z\right) + \beta\left|v\left(z\right)\right|^{p-2} v\left(z\right) \\ \qquad = \left(\theta\left(z\right) + \varepsilon\right)\left|v\left(z\right)\right|^{p-2} v\left(z\right) - \gamma_\varepsilon\left(z\right) \text{ a.e. on } Z, \\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right. \quad (78)$$

Arguing as in the proof of Proposition 51, we can find $\underline{v} \in -int\ C_+$, a solution of problem $\left(78\right)$. By virtue of $\left(77\right)$, we see that $\underline{v}$ is a strict lower solution for problem $\left(56\right)$. Then, truncating the nonlinearity $f\left(z,.\right)$ at the points $\left\{\underline{v}\left(z\right), 0\right\}$ and reasoning as in the proof of Proposition 51 we obtain:

Proposition 40 *If hypotheses* $\mathbf{H}\left(f\right)$ *hold, then problem* $\left(56\right)$ *admits a solution* $v_0 \in -int\ C_+$ *which is a local minimizer of* φ

Combining Propositions 51 and 52, we can summarize the results of this section in the following Theorem.

Theorem 41 *If hypotheses* $\mathbf{H}\left(f\right)$ *hold, then problem* $\left(56\right)$ *admits two constant sign smooth solutions* $x_0 \in int\ C_+$ *and* $v_0 \in -int\ C_+$*, which are local minimizers of the Euler functional* φ.

7 Degree theory

Degree theory is a basic tool of nonlinear analysis and produces powerful existence and multiplicity results for nonlinear boundary value problems.

It concerns operator equations of the form $\varphi(x) = y_0$, where φ is a map (often continuous) of $\overline{U}$, the closure of an open set U of the domain space X, into the range space Y and $y_0 \in Y$ satisfies $y_0 \notin \varphi(\partial U)$. Then the degree of S at y_0 relative to U, written $d(\varphi, U, y_0)$, is an algebraic count of the number of solutions of the equation

$$\varphi(x) = y_0.$$

In particular the equation $\varphi(x) = y_0$ will have solutions in U, whenever

$$d(\varphi, U, y_0) \neq 0.$$

Here we shall deal with degree theories, where the value of the degree map is an integer. This integer can be both positive or negative and in the context of finite dimensional problems positive counts (positive degree) correspond to solutions at which φ is orientation preserving, while negative counts correspond to solutions at which φ is not orientation preserving.

We start with a brief analytical presentation of Brouwer's degree theory, which is the first such theory and was introduced by Brouwer in 1912.

Definition 42 *Let $U \subseteq \mathbb{R}^N$ be a nonempty, bounded open set and $\varphi \in C^1(\overline{U}, \mathbb{R}^N)$. We say that $x \in \overline{U}$ is a critical point of φ, if*

$$J_\varphi(x) = \det \nabla\varphi(x) = 0, \ \textit{where } \nabla\varphi(x) = \left[\frac{\partial \varphi_i}{\partial x_j}(x)\right]_{i,j=1}^N.$$

Let

$$C_\varphi := \{x \in \overline{U} : J_\varphi(x) = 0\}.$$

The set $\varphi(C_\varphi)$ is called the crease *of φ. If $y \notin \varphi(C_\varphi)$, then y is called a regular value of φ.*

Definition 43 *If $\varphi \in C^1(\overline{U}, \mathbb{R}^N)$ and $y \notin [\varphi(C_\varphi) \cup \varphi(\partial U)]$, then the* degree of φ at y with respect to U *is defined by*

$$d(\varphi, U, y) = \sum_{x \in \varphi^{-1}(y)} sgn\,(J_\varphi(x)),$$

where $sgn\,(u) = 1$ if $u > 0$ and $sgn\,(u) = -1$ if $u < 0$.

Note that since y is not a critical value, the set $\varphi^{-1}(y)$ is discrete and so the summation in the previous Definition is finite.

The determinant $J_\varphi(x)$ is positive or negative according as φ is orientation preserving or orientation reversing at x.

Definition 44 *Let $U \subseteq \mathbb{R}^N$ be a bounded open set and $\varphi \in C(\overline{U}, \mathbb{R}^N)$ and $y \notin \varphi(\partial U)$, then we define $d_B(\varphi, U, y)$ to be equal to $d(\widehat{\varphi}, U, y)$, where $\widehat{\varphi} \in C^1(\overline{U}, \mathbb{R}^N)$ satisfies*

$$\|\varphi(x) - \widehat{\varphi}(x)\| < d\,(y, \varphi(\partial U)) \text{ for all } x \in \overline{U}.$$

In fact it is easy to check that in the above Definition we can choose $\widehat{\varphi} \in C^1(\overline{U}, \mathbb{R}^N)$ such that $y \notin \widehat{\varphi}(C_{\widehat{\varphi}})$.

This way we have concluded the definition of Brouwer's degree which has been defined for the class $C(\overline{U}, \mathbb{R}^N)$. In the next theorem we summarize the basic properties of Brouwer's degree.

Theorem 45 *If $U \subseteq \mathbb{R}^N$ is a bounded open set, $\varphi \in C(\overline{U}, \mathbb{R}^N)$ and $y \notin \varphi(\partial U)$ then:*

*(i) (*Normalization:*) $d_B(Id, U, y) = 1$ for all $y \in U$;*

*(ii) (*Aditivity with respect to domain:*) If U_1, U_2 are disjoint open subsets of U and $y \notin \varphi\left(\overline{U} \setminus (U_1 \cup U_2)\right)$, then*

$$d_B(\varphi, U, y) = d_B(\varphi, U_1, y) + d_B(\varphi, U_2, y);$$

*(iii) (*Homotopy invariance*): if $h\colon [0,1] \times \overline{U} \longrightarrow \mathbb{R}^N$ is a continuous map and $y \notin h(t, \partial U)$ for all $t \in [0,1]$, then $d_B(h(t,\cdot), U, y)$ is independent of $t \in [0,1]$;*

*(iv) (*Dependence on the boundary values:*) if $\widehat{\varphi} \in C(\overline{U}, \mathbb{R}^N)$ and $\varphi|_{\partial U} = \widehat{\varphi}|_{\partial U}$, then*

$$d_B(\varphi, U, y) = d_B(\widehat{\varphi}, U, y);$$

*(v) (*Excision property:*) if $K \subseteq \overline{U}$ is closed and $y \notin \varphi(K)$, then*

$$d_B(\varphi, U, y) = d_B(\varphi, U \setminus K, y);$$

*(vi) (*Continuity with respect to φ:*) if $\widehat{\varphi} \in C(\overline{U})$ and $\|\varphi - \widehat{\varphi}\|_\infty < d\,(y, \varphi(\partial U))$, then $d_B(\widehat{\varphi}, U, y)$ is defined and equals $d_B(\varphi, U, y)$;*

*(vii) (*Existence property:*) If $d_B(\varphi, U, y) \neq 0$ then $\varphi^{-1}(y) \neq \varnothing$.*

Suppose now that X is a reflexive Banach space. Then by the Troyanski renorming theorem (see Gasinski-Papageorgiou [**?**], p.911), we can equivalently renorm X so that both X and X^* are locally uniformly convex and with Fréchet differentiable norms. So, in what follows, we may assume that both X and X^* are locally uniformly convex. Hence, if $\mathcal{F} : X \to X^*$ is the duality map defined by

$$\mathcal{F}(x) = \left\{ x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \right\},$$

we have that $\mathcal{F}$ is a homeomorphism.

Definition 46 *An* operator *$A : X \to X^*$, which is single-valued and everywhere defined, is said to be* of type $(S)_+$, *if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and*

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

one has $x_n \to x$ in X.

Let U be a bounded open set in X and let $A:\overline{U}\rightarrow X^{*}$ be a demicontinuous operator of type $(S)_{+}$. Let $\{X_{\alpha}\}_{\alpha\in J}$ be the family of all finite dimensional subspaces of X and let A_{α} be the Galerkin approximation of A with respect to X_{α}, that is

$$\left\langle A_{\alpha}\left(x\right),y\right\rangle_{X_{\alpha}}=\left\langle A\left(x\right),y\right\rangle$$

for all $x\in\overline{U}\cap X_{\alpha}$ and all $y\in X_{\alpha}$. By $\left\langle \cdot,\cdot\right\rangle_{X_{\alpha}}$ we denote the duality brackets for the pair $(X_{\alpha},X_{\alpha}^{*})$.

Definition 47 *For $x^{*}\notin A\left(\partial U\right)$, the degree map $d_{(S)_{+}}\left(A,U,x^{*}\right)$ is defined by*

$$d_{(S)_{+}}\left(A,U,x^{*}\right)=d_{B}\left(A_{\alpha},U\cap X_{\alpha},x^{*}\right)$$

for X_{α} large enough (in the sense of inclusion), where Here d_{B} stands for the classical Brouwer degree map. If X is separable and A is bounded (maps bounded sets to bounded ones), then we can use only a countable subfamily $\{X_{n}\}_{n\geq1}$ of $\{X_{\alpha}\}_{\alpha\in J}$ such that $\overline{\underset{n\geq1}{\cup}X_{n}}=X$. More details on the degree map $d_{(S)_{+}}$ can be founded in Browder [14] and Skrypnik [50].

Definition 48 *Let $2^{X^{*}}\backslash\{\varnothing\}$ be the family of all nonempty subsets of Y^{*}. A multivalued map (or multifunction) $G:X\rightarrow2^{X^{*}}\backslash\{\varnothing\}$ is said to be upper semicontinuous (usc for short) if, for every closed set $C\subseteq X^{*}$, we have that*

$$G^{-}\left(C\right)=\{x\in X:G\left(x\right)\cap C\neq\varnothing\}$$

is closed in X.

The generalized subdifferential $x\rightarrow\partial\varphi\left(x\right)$ is an usc multifunction from X with the norm topology into X^{*} furnished with the w^{*}-topology.

Definition 49 *We say that a multifunction $G:X\rightarrow2^{X^{*}}\backslash\{\varnothing\}$ belongs in class (P), if it is usc, for every $x\in X$, $G\left(x\right)$ is closed, convex and for every bounded subset $A\subseteq X$, we have*

$$G\left(A\right)=\underset{x\in A}{\cup}G\left(x\right)$$

is relatively compact in X^{*}.

From Cellina [16] (see also Hu-Papageorgiou [31], p.106), we know that:

Theorem 50 *If $D\subseteq X$ is an open subset and if $G:D\rightarrow2^{X^{*}}\backslash\{\varnothing\}$ is an usc multifunction with closed and convex values, then given any $\varepsilon>0$, we there exists a continuous map $g_{\varepsilon}:D\rightarrow X^{*}$ (called approximate selection of G) such that*

$$g_{\varepsilon}\left(x\right)\in G\left(\left(x+B_{\varepsilon}\right)\cap D\right)+B_{\varepsilon}^{*}$$

for all $x\in D$ and $g_{\varepsilon}\left(D\right)\subseteq\overline{conv}\ G\left(D\right)$. Here

$$B_{\varepsilon}=\{x\in X:\left\Vert x\right\Vert<\varepsilon\}\ \text{and}\ B_{\varepsilon}^{*}=\{x^{*}\in X^{*}:\left\Vert x^{*}\right\Vert<\varepsilon\}.$$

Note that, if the multifunction G belongs in class (P), then the continuous approximate selection g_ε is compact.

Definition 51 *If $G : X \to 2^{X^*} \setminus \{\varnothing\}$ is a multifunction belonging in class (P) ,then for every $x^* \notin (A+G)(\partial U)$, $\widehat{d}(A+G,U,x^*)$ is defined by*

$$\widehat{d}(A+G,U,x^*) := d_{(S)_+}(A+g_\varepsilon,U,x^*)$$

for $\varepsilon > 0$ small, where g_ε is the continuous approximate selection of Ggiven by the previous Theorem.

Note that since G belongs in class (P), $g_\varepsilon : \overline{U} \to X^*$ is compact and so $x \longmapsto A(x) + g_\varepsilon(x)$ is still of type $(S)_+$. More about the degree map $\widehat{d}$, can be found in Hu-Papageorgiou [30], [31].

One of the fundamental properties of a degree map is the homotopy invariance property. To formulate this property for the degree map $\widehat{d}$, we need to define the admissible homotopies for A and G.

Definition 52 *(a) A one-parameter family $\{A_t\}_{t\in[0,1]}$ of maps from $\overline{U}$ into X^*, is said to be a* homotopy of class $(S)_+$, *if for any $\{x_n\}_{n\geq 1} \subseteq \overline{U}$ such that $x_n \xrightarrow{w} x$ and for any $\{t_n\}_{n\geq 1} \subseteq [0,1]$ with $t_n \to t$ for which*

$$\limsup_{n\to\infty} \langle A_{t_n}(x_n), x_n - x\rangle \leq 0,$$

one has $x_n \to x$ in X and $A_{t_n}(x_n) \xrightarrow{w} A_t(x_n)$ in X^* as $n \to \infty$.

(b) A one-parameter family $\{G_t\}_{t\in[0,1]}$ of multifunctions $G_t : \overline{U} \to 2^{X^*} \setminus \{\varnothing\}$ is said to be a homotopy of class (P), *if $(t,x) \longmapsto G(t,x)$ is usc from $T \times X$ into $2^{X^*} \setminus \{\varnothing\}$, for every $(t,x) \in [0,1] \times \overline{U}$ the set is closed, convex and*

$$\overline{\bigcup \left\{G_t(x) : t \in [0,1],\ x \in \overline{U}\right\}}$$

is compact in X^*.

With these admissible homotopies for A and G, the homotopy invariance property of $\widehat{d}$ can be formulated as follows:

> *"If $\{A_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$ such that for every $t \in [0,1]$, A_t is bounded, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $x^* : [0,1] \to X^*$ is a continuous map such that*
>
> $$x_t^* \notin (A_t + G_t)(\partial U) \text{ for all } t \in [0,1],$$
>
> *then $\widehat{d}(A_t + G_t, U, x_t^*)$ is independent of $t \in [0,1]$."*

Also the normalization property has the following form:

$$\widehat{d}(\mathcal{F},U,x^*) = d_{(S)_+}(\mathcal{F},U,x^*) = 1 \text{ for all } x^* \in \mathcal{F}(U).$$

Both degree maps $d_{(S)_+}$ and $\widehat{d}$ have all the usual properties mentioned for Briwer degree in Theorem 25, such as normalization, homotopy invariance, solution property, additivity with respect to the domain, excision property, etc. For further details and applications we refer to [2].

8 Degree theoretic approach

Consider again the problem from Section 7, where we proved the existence of two constant sign solutions by varitional method. In this section we want to empoy degree theory in order to prove the existence of the third soution under assumptions under assumptions $H(j)$.

So our problem is the following

$$\begin{cases} -\triangle_p x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ x \mid_{\partial Z} = 0,\ 1 < p < \infty, \end{cases} \tag{79}$$

and let $\varphi : W_0^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for problem (79) defined by

$$\varphi(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z j(z, x(z))\, dz \text{ for all } x \in W_0^{1,p}(Z).$$

Also for the multivalued Nemytskii operator $N : L^p(Z) \to 2^{L^{p'}(Z)}$ from Aizicovici-Papageorgiou-Staicu [2], we have:

Proposition 53 *If hypotheses $H(j)$ hold, then the multivalued operator $N : L^p(Z) \to 2^{L^{p'}(Z)}$ has nonempty, w-compact, convex values and it is usc from $L^p(Z)$ with the norm topology into $L^{p'}(Z)$ with the weak topology.*

From the Sobolev embedding theorem, we know that $W_0^{1,p}(Z)$ is embedded compactly and densely in $L^p(Z)$. It follows that $L^{p'}(Z) = L^p(Z)^*$ is embedded compactly and densely in $W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$. So from Proposition 45, we deduce the following:

Corollary 54 *If hypotheses $H(j)$ hold and $N = N|_{W_0^{1,p}(Z)} : W_0^{1,p}(Z) \to 2^{W^{-1,p'}(Z)} \setminus \{\varnothing\}$, then N is a multifunction of class (P).*

From Theorem 4 and its proof, we know that $x_0 \in int\ C_+$ (resp. $v_0 \in -int\ C_+$) is a minimizer of φ_+(resp. φ_-). Since $\varphi_+ \mid_{C_+} = \varphi$ (resp. $\varphi_- \mid_{C_-} = \varphi$) we infer that x_0, v_0 are both local $C_0^1\left(\overline{Z}\right) -$ minimizers of φ. But then from Gasinski-Papageorgiou [27], p. 655-656 (see also Kyritsi-Papageorgiou [34]), we have that x_0 and v_0 are local $W_0^{1,p}(Z)$ - minimizers of φ. Therefore from Aizicovici-Papageorgiou-Staicu [2], we have

Proposition 55 *If hypotheses $H(j)$ hold and $x_0 \in int\ C_+$, $v_0 \in -int\ C_+$ are the solutions obtained in Theorem 44, then there exists $r > 0$ small such that*

$$\widehat{d}(\partial\varphi, B_r(x_0), 0) = \widehat{d}(\partial\varphi, B_r(v_0), 0) = 1.$$

Next we calculate the $\widehat{d}$ degree of $\partial\varphi$ for small balls.

Proposition 56 *If hypotheses $H(j)$ hold, then there exists $\rho_0 > 0$ such that*

$$\widehat{d}(\partial\varphi, B_\rho, 0) = \widehat{d}(A - N, B_\rho, 0) = 0 \textit{ for all } 0 < \rho \leq \rho_0.$$

Proof. Let $K_+ : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the mapping defined by

$$K_+(x)(\cdot) = \left(x^+(\cdot)\right)^{p-1} \text{ for all } x \in W_0^{1,p}(Z).$$

Evidently, this is a completely continuous map (recall that $L^{p'}(Z)$ is embedded compactly into $W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$). So, if we consider the map $h_1 : [0,1] \times W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$h_1(t,x) = A(x) - (1-t)\eta_1 K_+(x) - tN(x) \text{ for all } (t,x) \in [0,1] \times W_0^{1,p}(Z),$$

then $h_1(.,.)$ is an admissible homotopy.

Claim: *There exists $\rho_0 > 0$ such that $0 \notin h_2(t,x)$ for all $t \in [0,1]$, all $\|x\| = \rho$ and all $0 < \rho \leq \rho_0$.*

We argue indirectly. So suppose that the Claim is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to 0 \text{ and } 0 \in h_1(t_n, x_n) \text{ for all } n \geq 1. \tag{80}$$

From the inclusion in (80), we have

$$A(x_n) = (1-t_n)\eta_1 K_+(x_n) + t_n u_n \text{ with } u_n \in N(x_n) \text{ for all } n \geq 1. \tag{81}$$

Setting

$$y_n = \frac{x_n}{\|x_n\|}$$

we have

$$A(y_n) = (1-t_n)\eta_1 K_+(y_n) + t_n \frac{u_n}{\|x_n\|^{p-1}} \text{ for all } n \geq 1.$$

Moreover, by passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z),\ y_n \to y \text{ in } L^p(Z),\ y_n(z) \to y(z) \text{ a.e. on } Z$$

and

$$|y_n(z)| \leq k(z) \text{ a.e. on } Z, \text{ for all } n \geq 1 \text{ and some } k \in L^{p'}(Z)_+.$$

By virtue of hypotheses $H(j)$ (iii), (iv) and (v), we have that

$$|u| \leq c_1 |x|^{p-1} \text{ for a.a. } z \in Z, \text{all } x \in \mathbb{R} \text{ and all } u \in \partial j(z,x), \tag{82}$$

with $c_1 > 0$. Relation (82) implies that

$$\left\{ \frac{u_n}{\|x_n\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

So we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(Z) \text{ for some } h_0 \in L^{p'}(Z). \tag{83}$$

For every $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$C^+_{\varepsilon,n} = \left\{ z \in Z : x_n(z) > 0,\, \eta_1(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)^{p-1}} \leq \eta_2(z) + \varepsilon \right\}$$

and

$$C^-_{\varepsilon,n} = \left\{ z \in Z : x_n(z) < 0,\ -\varepsilon \leq \frac{u_n(z)}{|x_n(z)|^{p-2}\, x_n(z)} \leq \varepsilon \right\}.$$

Note that $x_n(z) \to 0^+$ a.e. on $\{y > 0\}$ and $x_n(z) \to 0^-$ a.e. on $\{y < 0\}$. So, by virtue of hypothesis $H(j)(v)$, we have

$$\chi_{C^+_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y > 0\} \text{ and } \chi_{C^-_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

Using (83), we obtain

$$\chi_{C^+_{\varepsilon,n}} \frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(\{y > 0\})$$

and

$$\chi_{\chi_{C^-_{\varepsilon,n}}} \frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(\{y < 0\}).$$

From the definitions of the sets $C^+_{\varepsilon,n}$ and $C^-_{\varepsilon,n}$, we have

$$\begin{aligned}\chi_{C^+_{\varepsilon,n}}(z)(\eta_1(z) - \varepsilon)\, y_n(z)^{p-1} &\leq \chi_{C^+_{\varepsilon,n}}(z) \frac{u_n(z)}{\|x_n\|^{p-1}} \\ &\leq \chi_{C^+_{\varepsilon,n}}(z)(\eta_2(z) + \varepsilon)\, y_n(z)^{p-1} \text{ a.e. on } \{y > 0\}\end{aligned}$$

and

$$\begin{aligned}-\chi_{C^-_{\varepsilon,n}}(z)\, \varepsilon\, |y_n(z)|^{p-1} &\leq \chi_{C^-_{\varepsilon,n}}(z) \frac{u_n(z)}{\|x_n\|^{p-1}} \\ &\leq \chi_{C^-_{\varepsilon,n}}(z)\, \varepsilon\, |y_n(z)|^{p-1} \text{ a.e. on } \{y < 0\}.\end{aligned}$$

Taking weak limit in $L^{p'}(\{y > 0\})$ and $L^{p'}(\{y < 0\})$ respectively and using Mazur's lemma we obtain

$$(\eta_1(z) - \varepsilon)\, y(z)^{p-1} \leq h_0(z) \leq (\eta_2(z) + \varepsilon)\, y(z)^{p-1} \text{ a.e. on } \{y > 0\}$$

and

$$-\varepsilon\, |y(z)|^{p-1} \leq h_0(z) \leq \varepsilon\, |y_n(z)|^{p-1} \text{ a.e. on } \{y < 0\}.$$

Passing to the limit as $\varepsilon \downarrow 0$, yields

$$\eta_1(z)\, y(z)^{p-1} \leq h_0(z) \leq \eta_2(z)\, y(z)^{p-1} \text{ a.e. on } \{y > 0\} \tag{84}$$

and

$$h_0(z) = 0 \text{ a.e. on } \{y < 0\}. \tag{85}$$

Moreover, it is clear from (82) that

$$h_0(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{86}$$

From (84), (85) and (86) it follows that

$$h_0(z) = g_0(z)\, y^+(z)^{p-1} \text{ a.e. on } Z \tag{87}$$

with

$$g_0 \in L^\infty(Z)_+, \eta_1(z) \leq g_0(z) \leq \eta_2(z) \text{ a.e. on } Z.$$

Note that

$$\begin{aligned} &\langle A(y_n), y_n - y\rangle \\ &= \int_Z \left((1-t_n)\,\eta_1\left(y_n^+\right)^{p-1} + t_n \frac{u_n}{\|x_n\|^{p-1}}\right)(y_n - y)\, dz \to 0 \text{ as } n \to \infty. \end{aligned} \tag{88}$$

But A is of type $(S)_+$ (see Proposition 5). So from (87) it follows that

$$y_n \to y \text{ in } W_0^{1,p}(Z), \text{ hence } \|y\| = 1, \text{ i.e. } y \neq 0.$$

Moreover, in the limit as $n \to \infty$, we have

$$A(y) = \xi K_+(y)$$

where $\xi \in L^\infty(Z)_+$, $\xi = (1-t)\,\eta_1 + t g_0$. Hence acting with the test function $-y^- \in W_0^{1,p}(Z)$, we see that $y \geq 0$, $y \neq 0$. Therefore

$$\left\{\begin{array}{l} -div\left(\|Dy(z)\|^{p-2} Dy(z)\right) = \xi(z)\,|y(z)|^{p-2}\, y(z) \text{ a.e. on } Z, \\ y|_{\partial Z} = 0,\ y \neq 0. \end{array}\right. \tag{89}$$

Note that $\lambda_1 \leq \xi(z)$ a.e. on Z, $\lambda_1 \neq \xi$. Therefore

$$\widehat{\lambda}_1(\xi) < \widehat{\lambda}_1(\lambda_1) = 1. \tag{90}$$

Combining (89) and (90), we infer that $y \in C^1\left(\overline{Z}\right)$ must change sign, a contradiction to the fact that $y \geq 0$. So the Claim is true.
Then the homotopy invariance property, implies

$$\widehat{d}(A - N, B_\rho, 0) = \widehat{d}(A - \eta_1 K_+, B_\rho, 0) \text{ for all } 0 < \rho \leq \rho_0. \tag{91}$$

To compute $\widehat{d}(A - \eta_1 K_+, B_\rho, 0)$, we consider the homotopy $h_2 : [0,1] \times W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$h_2(t,x) = A(x) - \eta_1 K_+(x) - t\beta, \text{ for all } (t,x) \in [0,1] \times W_0^{1,p}(Z),$$

with $\beta \in L^\infty(Z)_+$, $\beta \neq 0$. Suppose that $h_2(t,x) = 0$ for all $t \in [0,1]$ and all $\|x\| = \rho$. Then

$$A(x) = \eta_1 K_+(x) + t\beta.$$

Acting with the test function $-x^{-} \in W_0^{1,p}(Z)$, we obtain $x \geq 0$. Hence

$$\begin{cases} -div\left(\|Dx(z)\|^{p-2} Dx(z)\right) = \eta_1(z)|x(z)|^{p-2} x(z) + t\beta(z) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0, \, x \neq 0 \end{cases}$$

and this by the antimaximum principle of Godoy-Gossez-Paczka [29] implies that $x \in -int\, C_{+}$, a contradiction to the fact that $x \geq 0$. So

$$\widehat{d}(A - \eta_1 K_{+}, B_\rho, 0) = \widehat{d}(A - \eta_1 K_{+} - \beta, B_\rho, 0) = 0 \text{ for all } 0 < \rho \leq \rho_0,$$

hence

$$\widehat{d}(A - N, B_\rho, 0) = 0 \text{ for all } 0 < \rho \leq \rho_0$$

(see (91)). ■

Next we conduct a similar computation for large balls. In this case we have:

Proposition 57 *If hypotheses $H(j)$ hold, then there exists $R_0 > 0$ such that*

$$\widehat{d}(\partial\varphi, B_R, 0) = \widehat{d}(A - N, B_R, 0) = 1 \textit{ for all } R \geq R_0.$$

Proof. We consider the admissible homotopy $h_3 : [0,1] \times W_0^{1,p}(Z) \to 2^{W^{-1,p'}(Z)}$ defined by

$$h_3(t,x) = A(x) - tN(x) - (1-t)\theta K(x)$$

where $K(x) = |x|^{p-2} x$.

Claim: *There exists $R_0 > 0$ such that $0 \notin h_3(t,x)$ for all $t \in [0,1]$ and all $x \in W_0^{1,p}(Z)$ with $\|x\| = R$ and all $R \geq R_0$.*

As in the previous proof, we argue by contradiction. So suppose we can find $\{t_n\}_{n\geq1} \subseteq [0,1]$ and $\{x_n\}_{n\geq1} \subseteq W_0^{1,p}(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to +\infty$$

and

$$A(x_n) = t_n u_n + (1-t_n)\theta K(x_n) \text{ with } u_n \in N(x_n), \ n \geq 1.$$

If

$$y_n = \frac{x_n}{\|x_n\|}, n \geq 1,$$

then

$$A(y_n) = t_n \frac{u_n}{\|x_n\|^{p-1}} + (1-t_n)\theta K(y_n). \tag{92}$$

Arguing as in the previous proof, using this time hypothesis $H(j)(iv)$, we obtain

$$y_n \to y \text{ in } W_0^{1,p}(Z), \text{ hence } \|y\| = 1,$$

and

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h = g|y|^{p-2} y \text{ with } g \in L^\infty(Z)_{+}, \ g(z) \leq \theta(z) \text{ a.e. on } Z.$$

So, if we pass to the limit as $n \to \infty$ in (92), we obtain

$$A(y) = \widehat{\xi} K(y) \text{ with } \widehat{\xi} \in L^{\infty}(Z), \ \widehat{\xi} = tg + (1-t)\theta \leq \theta,$$

hence

$$\left\{ \begin{array}{l} -div\left(\|Dy(z)\|^{p-2} Dy(z)\right) = \widehat{\xi}(z) |y(z)|^{p-2} y(z) \text{ a.e. on } Z, \\ y|_{\partial Z} = 0. \end{array} \right. \tag{93}$$

But

$$\widehat{\lambda}_1\left(\widehat{\xi}\right) > \widehat{\lambda}_1(\lambda_1) = 1.$$

So from (93) it follows that $y = 0$, a contradiction to the fact that $\|y\| = 1$. So the Claim is true.

Then the homotopy invariance property implies that

$$\widehat{d}(A - N, B_R, 0) = \widehat{d}(A - \theta K, B_R, 0) \text{ for all } R \geq R_0 \tag{94}$$

But from Drabek-Kufner-Nicolosi [22], we have

$$\widehat{d}(A - \theta K, B_R, 0) = 1 \text{ for } R > 0. \tag{95}$$

So from (94) and (95), we conclude that

$$\widehat{d}(A - N, B_R, 0) = 1 \text{ for all } R \geq R_0.$$

■

Now we are ready for the three solutions theorem for problem (79).

Theorem 58 *If hypotheses $H(j)$ hold, then problem* (79) *has at least three nontrivial solutions $x_0 \in int\, C_+$, $v_0 \in -int\, C_+$ and $y_0 \in C_0^1\left(\overline{Z}\right)$.*

Proof. We already have two solutions $x_0 \in int\, C_+$, $v_0 \in -int\, C_+$ from Theorem 4. Then from the domain additivity and excision properties of the degree we have

$$\begin{aligned} \widehat{d}(A - N, B_R, 0) = {} & \widehat{d}(A - N, B_\rho, 0) \\ & + \widehat{d}(A - N, B_r(x_0), 0) + \widehat{d}(A - N, B_r(v_0), 0) \\ & + \widehat{d}\left(A - N, B_R \backslash \left(\overline{B_\rho \cup B_r(x_0) \cup B_r(v_0)}\right), 0\right) \end{aligned} \tag{96}$$

with $R \geq R_0$ large and $r > 0$ and $0 < \rho \leq \rho_0$ small such that

$$\rho < R, \ B_r(x_0) \cap \overline{B_\rho} = \varnothing \text{ and } \overline{B}_r(x_0), \ \overline{B}_\rho(v_0) \subseteq B_R.$$

Then, using Propositions 47, 48 and 49, from (96), we obtain

$$-1 = \widehat{d}\left(A - N, B_R \backslash \left(\overline{B_\rho \cup B_r(x_0) \cup B_r(v_0)}\right), 0\right).$$

From the solution property, we obtain $y_0 \in B_R \backslash \left(\overline{B_\rho \cup B_r(x_0) \cup B_r(v_0)}\right)$, hence $y_0 \neq 0$, $y_0 \neq x_0$, $y_0 \neq v_0$ such that

$$A(y_0) = \widehat{u}_0 \text{ with } \widehat{u}_0 \in N(y_0),$$

hence

$$\begin{cases} -div\left(\|D(y_0)(z)\|^{p-2} D(y_0)(z)\right) = \widehat{u}_0(z) \text{ a.e. on } Z, \\ (y_0)|_{\partial Z} = 0, \end{cases}$$

therefore $y_0 \in C_0^1\left(\overline{Z}\right)$ is a nontrivial solution of (79), distinct from x_0 and v_0.
■

9 Elements of Morse theory: critical groups

Now, let us recall some basic notions and results from Morse theory, which we will need in the next section to produce a nontrivial smooth solution.

Let X be a Banach space and $\varphi \in C^1(X)$.

For every $c \in \mathbb{R}$, let $\varphi^c = \{x \in X : \varphi(x) \leq c\}$ be the *sublevel set of* φ *at* c, $K = \{x \in X : \varphi'(x) = 0\}$ be the *critical set of* φ and let $K_c = \{x \in K : \varphi(x) = c\}$ be the *critical set of* φ *at level* $c \in \mathbb{R}$.

Let Y be a subspace of a Hausdorff topological space V and let $n \geq 0$ be an integer. By $H_n(V, Y)$ we denote the n^{th}- singular homology group of the pair (V, Y) with integer coefficients.

If $x_0 \in X$ is an isolated critical point of φ with $\varphi(x_0) = c$, then the *critical groups of* φ *at* x_0, are defined by

$$C_n(\varphi, x_0) = H_n(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\}),\ n \geq 0,$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$.

By the excision property of the singular homology theory, we infer that the above definition of critical groups is independent of U (see Chang [17], and Mawhin-Willem [**?**]).

In what follows, we assume that φ satisfies the usual *PS-condition.* Namely, if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a sequence such that $|\varphi(x_n)| \leq M$ for some $M > 0$ and all $n \geq 1$, and $\varphi'(x_n) \to 0$ in X^*, then $\{x_n\}_{n\in\mathbb{N}}$ has a strongly convergent subsequence (see [17], p.20, [28], p.611, and [39], p. 81).

Assume that $-\infty < \inf \varphi(K)$ and let $c < \inf \varphi(K)$. Then, the *critical groups of* φ *at infinity* are defined by

$$C_n(\varphi, \infty) = H_n(H, \varphi^c) \text{ for all } n \geq 0,$$

(see Bartsch-Li [9]). The deformation lemma (see, for example, [**?**], p.21) implies that this definition is independent of the choice of c.

If $\varphi \in C^1(X)$ and $K = \{x_0\}$, then Morse theory implies that

$$C_n(\varphi, x_0) = C_n(\varphi, \infty) \text{ for all } n \geq 0.$$

In particular, if x_0 is an isolated critical point of φ and $C_n(\varphi, x_0) \neq C_n(\varphi, \infty)$ for some $n \geq 0$, then φ must have another critical point, distinct from x_0.

Moreover, if K is finite, then the *Morse type numbers of* φ are defined by

$$M_n = \sum_{x \in K} rank\ C_n(\varphi, x),\ n \geq 0,$$

and the *Betti-type numbers of* φ, are defined by

$$\beta_n = rank\ C_n(\varphi, \infty),\ n \geq 0.$$

By Morse theory (see Bartsch-Li [9], Chang [17], and Mawhin-Willem [39]), we have the *Poincaré-Hopf formula*

$$\sum_{n \geq 0} (-1)^n M_n = \sum_{n \geq 0} (-1)^n \beta_n. \tag{97}$$

The next result is useful in the computation of critical groups at infinity. It is related to Lemma 2.4 of Perera-Schechter [47], where X is a Hilbert space.

Proposition 59 *Let* $(X, \|.\|)$ *be a Banach space of dual* $(X^*, \|.\|_*)$ *and let* $(t, x) \to \varphi_t(x)$ *be a function in* $C^1([0,1] \times X)$, *such that* $x \to \varphi_t'(x)$ *and* $x \to \partial_t \varphi_t(x)$ *are both locally Lipschitz. (Here by* $\varphi_t'(x)$ *we denote the Frechet derivative of* $x \to \varphi_t(x)$ *and by* $\partial_t \varphi_t$ *the derivative of* $t \to \varphi_t(x)$.*) If we can find* $R > 0$ *such that*

$$\inf \{\|\varphi_t'(x)\|_* : t \in [0,1], \|x\| > R\} > 0 \tag{98}$$

and

$$\xi_R := \inf \{\varphi_t(x) : t \in [0,1], \|x\| \leq R\} > -\infty, \tag{99}$$

then for all $c < \xi_R$, *the set* φ_0^c *is homemorphic to a subset of* φ_1^c.

Proof. Note that by virtue of (98), for every $t \in [0,1]$, we have

$$K_t = \{x \in X : \varphi_t'(x) = 0\} \subseteq \overline{B}_R, \tag{100}$$

with $\overline{B}_R = \{x \in X : \|x\| \leq R\}$. Because of (100), and since by hypothesis $\varphi \in C^1([0,1] \times X)$, it follows (see, for example, [39], p.127) that there exists a pseudogradient vector field $\widehat{v} = (v_0, v) : [0,1] \times \left(X \backslash \overline{B}_R\right) \to [0,1] \times X$ corresponding to φ.

Recalling the construction of the pseudogradient vector field in Chang [**?**], p.19, we see that we can take $v_0(t, x) = \partial_t \varphi_t(x)$.

By definition, the map $(t, x) \to v_t(x)$ is locally Lipschitz and in fact, for every $t \in [0,1]$, v_t is a pseudogradient vector field corresponding to the function φ_t (see Chang [17], p.19). Hence, for every $(t, x) \in [0,1] \times \left(X \backslash \overline{B}_R\right)$, we have

$$\langle \varphi_t'(x), v_t(x) \rangle \geq \|\varphi_t'(x)\|_*^2, \tag{101}$$

where by $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X). The map $w : [0,1] \times (X \backslash \overline{B}_R) \to X$ given by

$$w_t(x) = -\frac{|\partial_t \varphi_t(x)|}{\|\varphi_t'(x)\|_*^2} v_t(x), \tag{102}$$

is well defined and locally Lipschitz. Because of (99), we can fix $c \in \mathbb{R}$,

$$c < \inf \{\varphi_t(x) : t \in [0,1], \|x\| \le R\}, \tag{103}$$

such that $\varphi_0^c \neq \varnothing$ or $\varphi_1^c \neq \varnothing$. (If no such c can be found, then

$$C_n(\varphi_0, \infty) = C_n(\varphi_1, \infty) = \delta_{n,0}\mathbb{Z}$$

and so we are done). Without any loss of generality, we may assume that $\varphi_0^c \neq \varnothing$ (the argument is similar if $\varphi_1^c \neq \varnothing$). Let $y \in \varphi_0^c$ and consider the Cauchy problem

$$\frac{d}{dt}\eta(t) = w_t(\eta(t)) \text{ for all } t \in [0,1], \; \eta(0) = y. \tag{104}$$

From the local existence theorem (see Gasinski-Papageorgiou [28], p.618), we know that **(??)**, admits a local flow $\eta(t)$. On account of **(??)**, **(??)** and **(??)** we have

$$\begin{aligned}
\frac{d}{dt}\varphi_t(\eta(t)) &= \left\langle \varphi_t'(\eta(t)), \frac{d}{dt}\eta(t) \right\rangle + \partial_t \varphi_t(\eta(t)) \\
&= \langle \varphi_t'(\eta(t)), w_t(\eta(t)) \rangle + \partial_t \varphi_t(\eta(t)) \\
&\le -|\partial_t \varphi_t(\eta(t))| + \partial_t \varphi_t(\eta(t)) \\
&\le 0.
\end{aligned}$$

Therefore, $t \to \varphi_t(\eta(t))$ is decreasing and so, we have

$$\varphi_t(\eta(t)) \le \varphi_0(\eta(0)) = \varphi_0(y) \le c$$

(recall that $y \in \varphi_0^c$).Because of **(??)**, we have that $\|\eta(t)\| > R$. Consequently, $\varphi_t'(\eta(t)) \neq 0$ and so the flow η is in fact global. Moreover, it can be reversed by replacing φ_t by φ_{1-t}. Therefore, $\eta(1)$ is a homeomorphism between φ_0^c and a subset of φ_1^c. ■

10 Multiple solutions by Morse theory

Consider again the problem (56), that is

$$\left\{ \begin{array}{l} -\triangle_p x(z) + \beta |x(z)|^{p-2} x(z) = f(z, x(z)) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right. \tag{105}$$

where $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2 boundary ∂Z and assumptions $\mathbf{H}(f)$ are satisfied. In this section, using Morse theory, we produce a third nontrivial smooth solution for problem (105). Note that the Euler functional φ satisfies the PS-condition, as one can easily verify.

In view of Theorem 53 and recalling the characterization of the critical group of a C^1- functional at a local minimizer (see Chang [17], p.33 and Mawhin-Willem [39], p.175), we have:

Proposition 60 *If hypotheses* $\mathbf{H}(f)$ *hold, then* $C_k(\varphi, x_0) = C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z}$ *for all* $k \geq 0$.

Next we compute the critical groups of φ at $x = 0$. Our approach is inspired by the semilinear works of Moroz [40] and Wang [52].

Proposition 61 *If hypotheses* $\mathbf{H}(f)$ *hold, then* $C_k(\varphi, 0) = 0$ *for all* $k \geq 0$.

Proof. By virtue of hypotheses $\mathbf{H}(f)(iii)$, (v) and (vi), we have

$$F(z,x) \geq c_1 |x|^r - c_2 |x|^p \text{ for a.a. } z \in Z \text{ and all } x \in \mathbb{R}, \tag{106}$$

with $c_1, c_2 > 0$. Then for $t > 0$ and $x \in W_n^{1,p}(Z)$, $x \neq 0$,

$$\begin{aligned} \varphi(tx) &= \frac{t^p}{p} \|Dx\|_p^p + \frac{t^p \beta}{p} \|x\|_p^p - \int_Z F(z, tx(z))\, dz \\ &\leq \frac{t^p}{p}\gamma_1 \|x\|^p + t^p c_2 \|x\|_p^p - t^r c_1 \|x\|_r^r \end{aligned} \tag{107}$$

with $\gamma_1 = \max\{\beta, 1\}$ (see (106)). Because $r < p$, from (107) it follows that there exists $t_0 = t_0(x) \in (0,1)$ such that

$$\varphi(tx) < 0 \text{ for all } t \in (0, t_0). \tag{108}$$

Next we show that for every $x \neq 0$

$$\frac{d}{dt}\varphi(tx) > \frac{p}{t}\varphi(tx) \text{ for all } t > 0. \tag{109}$$

To this end, we remark that

$$\begin{aligned} \frac{d}{dt}\varphi(tx) &= \langle \varphi'(tx), x\rangle \\ &= \langle A(tx), x\rangle + \beta t^{p-1} \int_Z |x|^p\, dz - \int_Z f(z,tx)\, x dz \\ &= t^{p-1}\left(\|Dx\|_p^p + \beta\|x\|_p^p\right) - \frac{1}{t}\int_Z f(z,tx)\, tx dz \\ &= \frac{p}{t}\left[\frac{t^p}{p}\left(\|Dx\|_p^p + \beta\|x\|_p^p\right) - \frac{1}{p}\int_Z f(z,tx)\, tx dz\right] \\ &> \frac{p}{t}\left[\frac{t^p}{p}\left(\|Dx\|_p^p + \beta\|x\|_p^p\right) - \int_Z F(z,tx)\, dz\right] \text{ (see } \mathbf{H}(f)(vi)) \\ &= \frac{p}{t}\varphi(tx), \end{aligned}$$

which proves (109) .

We assume that the origin is an isolated critical point of φ, or otherwise we have a whole sequence of distinct solutions of (105), and so, we are done. Let $\rho > 0$ be small such that $K \cap B_\rho = \{0\}$, where $K = \left\{x \in W_n^{1,p}(Z) : \varphi'(x) = 0\right\}$ and $B_\rho = \left\{x \in W_n^{1,p}(Z) : \|x\| < \rho\right\}$.

We show that for any $x \in \varphi^0 \cap B_\rho$, we have $tx \in \varphi^0 \cap B_\rho$ for all $t \in [0,1]$ (recall that $\varphi^0 = \left\{x \in W_n^{1,p}(Z) : \varphi(x) \leq 0\right\}$). We argue indirectly. So, suppose that for some $t_0 \in (0,1)$, we have $\varphi(t_0 x) > 0$. Then, by continuity, there exists $t_1 \in (t_0, 1]$ such that $\varphi(t_1 x) = 0$. We take $t_1 = \min\{t \in [t_0,1] : \varphi(tx) = 0\}$. Hence, $\varphi(tx) > 0$ for all $t \in [t_0, t_1)$ and so

$$\frac{d}{dt}\varphi(tx)\mid_{t=t_1} \leq 0. \tag{110}$$

From (109) and (110) , we have

$$0 = \frac{p}{t_1}\varphi(t_1 x) < \frac{d}{dt}\varphi(tx)\mid_{t=t_1} \leq 0,$$

a contradiction. This proves that for all $x \in \varphi^0 \cap B_\rho$ and all $t \in [0,1]$, $tx \in \varphi^0 \cap B_\rho$. Therefore, for every $t \in [0,1]$, the map $x \to h(t,x) = (1-t)x$ maps $\varphi^0 \cap B_\rho$ into itself. Clearly, $(t,x) \to h(t,x)$ is continuous and $h(0,x) = x$ for all $x \in \varphi^0 \cap B_\rho$. Hence h is a continuous deformation of $\varphi^0 \cap B_\rho$ to itself and so, we conclude that $\varphi^0 \cap B_\rho$ is contractible into itself.

Next, we show that $\left(\varphi^0 \cap B_\rho\right) \setminus \{0\}$ is contractible in itself. For this purpose, we introduce the map $T : B_\rho \setminus \{0\} \to (0,1]$ by

$$T(x) = \begin{cases} 1 & \text{if} \quad x \in \left(\varphi^0 \cap B_\rho\right) \setminus \{0\} \\ t & \text{if} \quad x \in B_\rho \setminus \{0\} \text{ with } \varphi(tx) = 0, t \in (0,1) . \end{cases}$$

From (108) and (109) it is clear that the map T is well defined and, if $\varphi(x) > 0$, then there exists a unique $T(x) \in (0,1)$ such that $\varphi(tx) < 0$ for all $t \in (0, T(x))$, $\varphi(T(x)x) = 0$ and $\varphi(tx) > 0$ for all $t \in (T(x), 1]$. Also, we have

$$\frac{d}{dt}\varphi(tx)\mid_{t=T(x)} > \frac{p}{T(x)}\varphi(T(x)x) = 0 \text{ (see (109))}.$$

Invoking the implicit function theorem, we infer that $x \to T(x)$ is continuous. Let $\widehat{h} : B_\rho \setminus \{0\} \to \left(\varphi^0 \cap B_\rho\right) \setminus \{0\}$ be defined by

$$\widehat{h}(x) = \begin{cases} T(x)x & \text{if} \quad x \in B_\rho \setminus \{0\} , \ \varphi(x) \geq 0 \\ x & \text{if} \quad x \in B_\rho \setminus \{0\} , \ \varphi(x) < 0. \end{cases}$$

The continuity of T implies the continuity of $\widehat{h}$ (note that $T(x) = 1$ for all $x \in B_\rho \setminus \{0\}$ with $\varphi(x) = 0$). Clearly $\widehat{h}\mid_{\varphi^0 \cap B_\rho} = id\mid_{\varphi^0 \cap B_\rho}$, hence $\widehat{h}$ *is* a retraction and so $\left(\varphi^0 \cap B_\rho\right) \setminus \{0\}$ is a retract of $B_\rho \setminus \{0\}$. Because $W_n^{1,p}(Z)$ is infinite dimensional, $B_\rho \backslash \{0\}$ is contractible in itself. Recall that retracts of contractible spaces are contractible too. Therefore, we infer that $\left(\varphi^0 \cap B_\rho\right) \setminus \{0\}$ is contractible in itself. Consequently, from Mawhin-Willem [39], p.172, we have

$$C_k(\varphi, 0) = H_k\left(\varphi^0 \cap B_\rho, \left(\varphi^0 \cap B_\rho\right) \setminus \{0\}\right) \text{ for all } k \geq 0.$$

■

Next, using Proposition 41, we will compute the critical groups at infinity for the functional φ. Here we will need the restriction $p \geq 2$.

Proposition 62 *If hypotheses* $\mathbf{H}(f)$ *hold and* $2 \leq p < \infty$, *then*

$$C_k(\varphi, \infty) = \delta_{k,0}\mathbb{Z} \textit{ for all } k \geq 0.$$

Proof. We consider the functions

$$(t, x) \rightarrow \varphi_t(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\beta}{p}\|x\|_p^p - (1-t)\int\limits_Z F(z, x(z))\, dz,$$

for all $(t, x) \in [0,1] \times W_n^{1,p}(Z)$. Clearly $x \rightarrow \partial_t \varphi_t(x)$ is locally Lipschitz. Also $\varphi_t'(x) = Ax + \beta K_p(x) - tN(x)$. Since we assume $2 \leq p < \infty$, we see that $x \rightarrow \varphi_t'(x)$ is locally Lipschitz too. In order to apply Proposition 41 we need to verify (98) and (99). Clearly, (99) holds. So, it remains to check (98). We proceed by contradiction. So, suppose that (98) is not true. Then we can find sequences $\{t_n\}_{n\geq 1} \subseteq [0,1]$, $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ such that

$$t_n \rightarrow t, \ \ \|x_n\| \rightarrow \infty \text{ and } \varphi_{t_n}'(x_n) \rightarrow 0 \text{ in } W_n^{1,p}(Z)^*.$$

Then

$$\left|\left\langle \varphi_{t_n}'(x_n), u\right\rangle\right| \leq \varepsilon_n \|u\| \text{ for all } u \in W_n^{1,p}(Z), \text{ with } \varepsilon_n \downarrow 0.$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_n^{1,p}(Z) \text{ and } y_n \rightarrow y \text{ in } L^p(Z).$$

We have

$$\begin{aligned} &\left|\langle A(y_n), u\rangle + \beta \int\limits_Z |y_n|^{p-2} y_n u dz - (1-t_n)\int\limits_Z \tfrac{N(x_n)}{\|x_n\|^{p-1}} u dz\right| \\ &\qquad\qquad \leq \varepsilon_n \|u\| \text{ for all } u \in W_n^{1,p}(Z). \end{aligned} \tag{111}$$

Hypotheses $\mathbf{H}(f)(iii), (iv)$ imply that $\left\{\frac{N(x_n)}{\|x_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(Z)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ is bounded. So, setting $u = y_n - y$ in (111), we have

$$\beta\int\limits_Z |y_n|^{p-2} y_n (y_n - y)\, dz \rightarrow 0 \text{ and } \int\limits_Z \frac{N(x_n)}{\|x_n\|^{p-1}}(y_n - y)\, dz \rightarrow 0 \text{ as } n \rightarrow \infty.$$

From (111) it follows that

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0.$$

Invoking the fact that the nonlinear operator $A : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Dy)_{\mathbb{R}^N}\, dz \text{ for all } x,\, y \in W_n^{1,p}(Z).$$

is bounded, continuous, monotone and of type $(S)_+$ (see, e.g., [2]), we have that

$$y_n \to y \text{ in } W_n^{1,p}(Z), \tag{112}$$

hence

$$\|y\| = 1 \text{ and so } y \neq 0. \tag{113}$$

Reasoning as in the proof of Proposition 14 in Aizicovici-Papageorgiou-Staicu [2], we can show that

$$h_n = \frac{N(x_n)}{\|x_n\|^{p-1}} \xrightarrow{w} h \text{ in } L^{p'}(Z), \text{ with } h = g\,|y|^{p-2}\,y,\ g \in L^\infty(Z)_+,\ g \leq \theta. \tag{114}$$

Passing to the limit as $n \to \infty$ in (111) and using (112) and (113), we obtain

$$\langle A(y), u\rangle + \beta \int_Z |y|^{p-2}\, yudz = (1-t)\int_Z g\,|y|^{p-2}\, yudz. \tag{115}$$

Since $u \in W_n^{1,p}(Z)$ is arbitrary, from (115) it follows that

$$A(y) + \beta K_p(y) = (1-t)\, gK_p(y).$$

Because $tg \leq \theta$, using Lemma 1, we have

$$\widehat{\xi}_0\, \|y\|^p \leq 0, \text{ hence } y = 0,$$

a contradiction to (**??**). Therefore (**??**) holds for some $R > 0$. Applying Proposition 2, we can say that for $c < \xi_R$, φ_0^c is homeomorphic to a subset of φ_1^c. But note that by virtue of hypothesis $\mathbf{H}(f)(vi)$, $\varphi_0 \leq \varphi_1$, hence $\varphi_1^c \subseteq \varphi_0^c$. Therefore, φ_0^c and φ_1^c are homeomorphic, and so

$$C_k(\varphi_0, \infty) = C_k(\varphi_1, \infty) \text{ for all } k \geq 0. \tag{116}$$

Note that

$$\varphi_0(x) = \varphi(x) \text{ and } \varphi_1(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\beta}{p}\|x\|_p^p \text{ for all } x \in W_n^{1,p}(Z).$$

Clearly, φ_1 has only one critical point $x = 0$ and it is a global minimizer. Hence

$$C_k(\varphi_1, \infty) = C_k(\varphi_1, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{117}$$

Since $\varphi_0 = \varphi$, from (116) and (117), we conclude that

$$C_k(\varphi, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$

■

Now we are ready for the *three solutions theorem* for problem (**??**).

Theorem 63 *If hypotheses* $\mathbf{H}(f)$ *hold and* $2 \leq p < \infty$, *then problem* (**??**) *has at least three nontrivial smooth solutions* $x_0 \in int\ C_+$, $v_0 \in -int\ C_+$ *and* $y_0 \in C_n^1(\overline{Z})$.

Proof. From Theorem 1, we already have two nontrivial smooth solutions of constant sign, namely $x_0 \in int\ C_+$ and $v_0 \in -int\ C_+$. Suppose that 0, x_0 and v_0 are the only critical points of φ. Then from the Poincaré-Hopf fomula (see (**??**)) and Propositions 7, 8 and 9, we have

$$(-1)^0 + (-1)^0 = (-1)^0$$

hence $(-1)^0 = 0$, a contradiction. This shows that there must be a fourth critical point $y_0 \in W_n^{1,p}(Z)$ of φ, distinct from 0, x_0 and v_0. Evidently, y_0 is a solution of (**??**), and as before, the nonlinear regularity theory implies that $y_0 \in C_n^1(\overline{Z})$. ■

Remark 64 *In fact, with some additional effort, our work can be extended to the case when in* (**??**), *the p-Laplacian is replaced by a more general operator of the form div* $a(z, Dx(z))$, *with* $a(z,y) = D_y G(z,y)$, *where* $G : Z \times \mathbb{R}^N \to \mathbb{R}$ *is measurable in* $z \in Z$, *of class* C^1 *and convex in* $y \in \mathbb{R}^N$, *and satisfies (for all* $z \in Z$, $y \in \mathbb{R}^N$*)*

$$(a(z,y), y)_{\mathbb{R}^N} \leq p\ G(z,y) \text{ and } G(z,y) \geq c\|y\|^p \text{ for some } c > 0.$$

References

11 Multivalued analysis: some complementat topics

Control systems and differential inclusions

Consider a controlled dynamical system

$$x' = f(t,x,u),\ u(t) \in U \tag{118}$$

where $U \subset \mathbb{R}^m$ is a given set of control values and the the control of the system is done by choosing $u(.)$ in a class of admissible controls

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m : u \text{ is measurable and } u(t) \in U \text{ a.e. }\},$$

For every (t, x) define the set

$$F(t, x) = \{f(t, x, \omega) : \omega \in U\} \tag{119}$$

and remark that at a given time t and a given state $x(t)$, several velocities are possible: $x'(t)$ has to belong to the set $F(t, x(t))$. So what really counts for the dynamics, is this set, which can be described by a set-valued map $F : \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$, and the controlled differential equation becomes a differential inclusion

$$x' \in F(t, x). \tag{120}$$

Remark 65 *If* $x : I \to \mathbb{R}^n$ *is a trajectory of the system* (118), *(that is,* x *is absolutely continuous in* I *for which there exists an admissible control* $u(.) \in \mathcal{U}$ *such that* $x'(t) = f(t, x(t), u(t))$ *a.e.), then* $x(.)$ *is a solution (or trajectory) of the associated differential inclusion* (120), *that is*

$$x'(t) \in F(t, x(t)) \;\; a.e. \;\; t \in I. \tag{121}$$

Moreover, by Filippov's lemma (1962) any solution of (120) is also a trajectory of (118), therefore (118) and (120) have the same trajectories. This has led to the simplification of some proofs on the existence of optimal controls.

Although some papers on multivalued differential equations, or differential inclusions, appeared in the literature before the middle of the century (Zaremba (1936) and Marchaud (1938)) the subject began to interest mathematicians seriously in the 1960s. Several motivations concurred, but one of them was the interest in control theory and in optimal control.

What we obtained in (120) is an example of *ordinary differential inclusion* that is a relation of the kind

$$x' \in F(t, x). \tag{122}$$

There are also the so called *gradient inclusions,*

$$\nabla u(x) \in F(x, u(x)) \tag{123}$$

In the first case, one seeks a function $x(.)$, in general in the class of absolutely continuous functions, such that, for almost every t in some interval, the derivative $x'(t)$ exists and is contained in the set $F(t, x)$. In addition, initial or boundary conditions may be prescribed.

In the second case, an open region Ω is given and one seeks a function u in some Sobolev space $W^{1,p}(\Omega)$ whose Sobolev gradient, for almost every $x \in \Omega$, is contained in $F(x, u(x))$. In addition, auxiliary conditions, in general in the form of Dirichlet boundary conditions, are prescribed.

Ordinary differential inclusions

Here we will refer only the ordinary differential inclusions. Let $2^{\mathbb{R}^n}$ be the family of nonempty subsets of $\mathbb{R}^n$.

Definition 66 $F : \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *which assign to each* $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ *a unique set* $F(t, x) \subset \mathbb{R}^n$ *is said to be a set-valued map, or multivalued map or multifunction.*

To such a multifunction we associate the ordinary differential inclusion

$$x' \in F(t,x). \tag{124}$$

Definition 67 *By solution of* (124) *we mean an absolutely continuous function* $x : I \to \mathbb{R}^n$ *defined in some interval* I *such*

$$x'(t) \in F(t, x(t)) \;\; a.e. \;\; t \in I. \tag{125}$$

Differential inclusions provide tools for the study of (discontinuous or implicit) differential equations. the equation $f(t,x,x') = 0$ can be regarded as a differential inclusion(124) with

$$F(t,x) = \{v : f(t,x,v) = 0\}.$$

Existence of solutions

A first important problem was the existence of solutions. For this we need some assumptions on F which are essentially of two different kinds:

- continuity is some sense to be specified of the multivalued map F (upper semicontinuity, lower semicontinuity, continuity, Hausdorff continuity, Lipschitzianity, etc.)
- geometric or topological properties of the images of the map F (these images can be closed sets, convex sets, compact or unbounded, or can have nonempty interior).

For the case of an inclusion, showing the existence of a solution is (considerably) more difficult than showing the existence of a solution for the case of an ordinary differential equation $x'(t) = f(t, x(t))$ where f is continuous with respect to x.

This fact is contrary to intuition. In fact, shouldn't it be that, proving that the point x' equals the point $f(t,x)$, is more difficult than merely proving that it is in a set $F(t,x)$?

Multivalued maps

Definition 68 *Let* X *and* Y *be two sets. We call multivalued map or set-valued map or multifunction from* X *into* Y *an application* F *which assign to every* $x \in X$ *an unique subset of* Y, *denoted* $F(x)$ *and called the value of* F *in* x.

Denoting by 2^Y the family of all subsets of Y, we remark that a multivalued map from X into Y is a map from X into 2^Y that we denote by $F : X \to 2^Y$.

On the space 2^Y or some of its subspace we can consider some topology that allow to look to a multivalued map $F : X \to 2^Y$ as a ordinary map between the space X (at least topological space) and the topological space space 2^Y.

Hausdorff distance and Hausdorff continuity

Let (Y, d_Y) be a metric space and $\mathcal{C}(Y)$ be the space of closed bounded subsets of Y. The *Hausdorff distance* on $\mathcal{C}(Y)$ is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A} d_Y(a,B), \sup_{b\in B} d_Y(b,A)\right\}, \text{ for all } A,B \in \mathcal{C}(Y).$$

where $d_Y(a,B) = \inf_{b\in B} d_Y(a,b)$ is the distance from $a \in Y$ to $B \in \mathcal{C}(Y)$. Equivalently,

$$d_H(A,B) = \inf\{\rho > 0 : A \subset B(B,\rho) \text{ and } B \subset B(A,\rho)\}$$

where $B(A,\rho)$ is the $\rho-$*neighborhood around* A defined by

$$B(B,\rho) = \{x \in Y : d_Y(x,A) < \rho.\}$$

Definition 69 *Let X be a topological space and (Y, d_Y) a metric space. A multifunction $F : X \to \mathcal{C}(Y)$ is said to be Hausdorff continuous if for every $x_0 \in X$ and every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that*

$$d_H(F(x), F(x_0)) < \varepsilon \text{ for every } x \in U.$$

Upper and lower semicontinuity

Definition 70 *Let X and Y be topological spaces. A multivalued map $F : X \to 2^Y$ is said to be:*

(a) lower semicontinuous at $x_0 \in X$ if for every $y_0 \in F(x_0)$ and every V neighborhood of y_0 there exists U a neighborhood of x_0 such that: $x \in U$ implies $F(x) \cap V \neq \varnothing$;

(b) upper semicontinuous at $x_0 \in X$ if for every open set $V \subset Y$ containing $F(x_0)$ there exists a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$;

(c) continuous at x_0 if it is both upper and lower semicontinuous at x_0

(d) lower semicontinuous (upper semicontinuous, continuous) on X if it is lower semicontinuous (upper semicontinuous, continuous) at each $x_0 \in X$.

$F : X \to 2^Y$ is l.s.c. on X iff $F^+(C) := \{x \in X : F(x) \subset C\}$ is closed in X for every closed set $C \subset Y$;

$F : X \to 2^Y$ is u.s.c. on X iff $F^-(C) := \{x \in X : F(x) \cap C \neq \varnothing\}$ is closed in X for every closed set $C \subset Y$.

Relations and properties

All these definitions coincide with the usual continuity when F is single valued.

If $F : X \to 2^Y$ has compact values then F is continuous on X if and only if it is Hausdorff continuous

The graph of a multivalued map $F : X \to 2^Y$ is the set

$$graph(F) = \{(x,y) \in X \times Y : y \in F(x)\}.$$

Definition 71 *We say that $F : X \to 2^Y$ has closed graph if for every $x_0 \in X$ and every sequence $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ converging to x_0 and y_0, respectively and such that $y_n \in F(x_n)$ for any $n \in N$, one has that $y_0 \in F(x_0)$.*

Remark 72 *If $F : X \to 2^Y$ is upper semicontinuous on X with closed values then it has closed graph;*

Conversely, if F has closed graph and there exists a compact set $K \subset Y$ such that $F(x) \subset K$ for every $x \in X$ then F is upper semicontinuous on X.

Example 73 *$F_1 : \mathbb{R} \to 2^{\mathbb{R}}$ defined by*

$$F_1(x) = \begin{cases} \{1\} & \text{if } x < 0 \\ \{-1, 1\} & \text{if } x = 0 \\ \{-1\} & \text{if } x > 0 \end{cases}$$

is upper semicontinuous at $x_0 = 0$ (on $\mathbb{R}$).

Indeed, if V is an open set containing $F(0) = \{-1, 1\}$ then for $\delta > 0$ small enough and for every $x \in (-\delta, \delta)$ one has that $F_1(x) \subset V$.

F_1 is not l.s.c. at $x_0 = 0$ because taking $y_0 = 1 \in F(0)$ and $V = \left(\frac{1}{2}, \frac{3}{2}\right)$ neighborhood of $y_0 = 1$, for every $\delta > 0$ there exists $\widehat{x} \in (0, \delta)$ where $F_1(\widehat{x}) \cap V = \{-1\} \cap \left(\frac{1}{2}, \frac{3}{2}\right) = \varnothing$.

Example 74 *$F_2 : \mathbb{R} \to 2^{\mathbb{R}}$ defined by*

$$F_2(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ [-1, 1] & \text{if } x \neq 0 \end{cases}$$

is l.s.c. at $x_0 = 0$ but it is not u.s.c. at $x_0 = 0$. Indeed for every V neighborhood of $y_0 = 0 \in F_2(0)$ there exists $\delta > 0$ such that for every $x \in (-\delta, \delta)$ one has that $F_2(x) \cap V \neq \varnothing$, hence F_2 is l.s.c. at $x_0 = 0$.

F_2 is not upper at $x_0 = 0$ because $V = \left(-\frac{1}{2}, \frac{1}{2}\right)$ contains $F_2(0)$ and for every $\delta > 0$ and any $\widehat{x} \in (0, \delta)$, $F_2(\widehat{x})$ is not contained in V, hence F is not u.s.c. at $x_0 = 0$.

Existence results

Simplest case: F upper semicontinuous with compact convex values (Zaremba (1936), Marchaud (1938), Filippov (1963), Wazewski (1961);

Hermes question (1970): prove existence for the differential inclusion $x' \in F(t, x)$ .for F Hausdorff continuos with compact valued

Answers: Filippov (1972) for F continuous in both variables: Kaczinski-Olech (1974) and Antosiewicz-Cellina (1975) for F Carathéodory (measurable with respect to t and continuous with respect to x);

For F lower semicontinuous: Bressan (1980), Lojasiewicz Jr, (1980).

For F upper semicontinuous without convex values: no solution in general: for $F(t, x) = -1$ for $x < 0$, $F(t, x) = \{-1, 1\}$ if $x = 0$, $F(t, x) = 1$ for $x > 0$ and for $G(t, x) = -F(t, x)$ one has that $x' = G(t, x)$, $x(0) = 0$, $t \geq 0$ has no solutions and $x' = F(t, x)$, $x(0) = 0$, $t \geq 0$ has exactly two solutions.

The approximate solutions method for equations

Let x_n be a sequence of approximate solutions, of the equation $x' = f(t, x)$ such that

$$x_n'(t) - f(t, x_n(t)) = \varepsilon_n,$$

with $\varepsilon_n \to 0$, and let us assume that $(x_n)_n$ converge to a limit x_*.

If f is continuous in the variable x, we have that

$$x_n'(t) = f(t, x_n(t)) + \varepsilon_n \to f(t, x(t)).$$

Hence, from the fact that the approximate solutions x_n converge, one obtains that their derivatives converge and, passing to the limit, that

$$x'(t) = f(t, x(t)).$$

In the case of inclusions, knowing that x_n converge, even when the set-valued map F is very regular (for instance, constant!) there is no reason to think that their derivatives should converge (consider the case $F(t.x) = \mathbb{R}^N$.).

The approximate solutions method for inclusions

For the differential inclusion $x' \in F(t,x)$ we construct a sequence of approximate solutions $(x_n)_{n\geq 1}$ in the sense that there exists a sequence $(\varepsilon_n)_{n\geq 1}$ converging uniformly to zero such that

$$d(x_n'(t), F(t, x_n(t) - \varepsilon_n(t))) \to 0 \tag{126}$$

and show that a subsequence converge to a solution.

The convergence of $(x_n)_{n\geq 1}$ to some $\widehat{x}$ in the space of absolutely continuous functions implies the weak convergence in L^1 of the derivatives x_n' to $\widehat{x}'$, and, the uniformly convergence of $(x_n)_{n\geq 1}$, and $(\varepsilon_n)_{n\geq 1}$ and the relation (126) imply

$$\widehat{x}_n'(t) \in \bigcap_{\varepsilon>0} \overline{co} \bigcup_{\|x-\widehat{x}(t)\|<\varepsilon} F(t,x) = \varepsilon_n, \tag{127}$$

where $\overline{co}$ denotes the closed convex hull.

If the right-hand side of (127) is $F(t, \widehat{x}(t))$ (what happens if $F(t,.)$ is upper semicontinuous, compact convex valued) then $\widehat{x}(.)$ is a solution of $x' \in F(t,x)$.

Through a very clever and simple construction, Filippov was able to build a sequence of approximate solutions whose derivatives bounded in L^∞ were piecewise constant maps and equi oscillating.

Relation with Tonelli's weak lower semicontinuity

The standard proof of the existence to differential inclusions with u.s.c. right-hand side is essentially the same proof as that of Tonelli's weak lower semicontinuity theorem, in the Calculus of Variations, where one wants to minimize

$$\int_a^b L(t, x(t), x'(t))\, dt. \tag{128}$$

One considers $(x_n)_n$, a minimizing sequence, in the case of the minimum problem, or a sequence of approximate solutions, for the case of a differential inclusion.

From some a priory estimates for the solutions to the differential inclusion, or from the coercivity assumptions for the functional one is trying to minimize, one infers that the derivatives x_n are weakly pre-compact in L^1 (or in L^p);

An application of Mazur's Lemma yields the strong convergence of a sequence of convex combinations of x'_n.

At this point, exploiting either the convexity of the values of F, or the convexity of L with respect to the variable x',, one obtains the result.

The fixed point approach for equations

To prove existence of solutions to a Cauchy problem

$$x' = f(t, x), x(t_0) = x_0 \tag{129}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded by a constant M then define the set

$$K = \{x \in C(I, \mathbb{R}^n) : x(0) = x_0, \|x(t) - x(s)\| \leq M|t - s| \ \ \forall t, s \in I\}$$

and the Picard operator

$$T(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Then K is compact and convex, T is continuous and $T(K) \subset K$, hence by Schauder fixed point theorem, there exists a fixed point $\overline{x} \in K$, and this is a solution of (7).

The fixed point approach for inclusions

If F is a multivalued map with compact valued contained in a ball $B(0, M)$, the analog of Picard operator is the multifunction $\mathcal{T} : K \to 2^K$ defined by

$$\mathcal{T}(x) = \{z \in K : z'(t) \in F(t, x(t))\, ds \text{ a.e. } t \in I\}$$

and we remark that any fixed point of $\mathcal{T}(x)$ (i.e., such that $x \in \mathcal{T}(x)$) is a solution of the Cauchy problem

$$x' \in F(t, x), x(t_0) = x_0 \tag{130}$$

If F is upper semicontinuous with compact convex values, the same holds for $\mathcal{T}(x)$, and by Kakutani fixed point theorem, $\mathcal{T}$ has a fixed point. Then problem (130) has solutions, that we already proved by direct method.

Let consider the nonconvex case, and remark first that if $h : K \to K$ is a continuous selection of $\mathcal{T}$, then by applying Schauder fixed point there exists $z \in K$ a fixed point of h, which is a solution of (130).

The fixed point approach for inclusions

Let $\Phi : K \to 2^{L^1(I, \mathbb{R}^n)}$ be defined by

$$\Phi(x) = \left\{u \in L^1(I, \mathbb{R}^n) : u(t) \in F(t, x(t))\, ds \text{ a.e. } t \in I\right\}. \tag{131}$$

If $g : K \to L^1(I, \mathbb{R}^n)$ is a continuos selection of Φ then $h : K \to K$ defined by

$$h(x)(t) = x_0 + \int_{t_0}^{t} g(x)(s)\, ds$$

is a continuous selection of $\mathcal{T}$, and so the existence of solutions is proved provided such a continuous selection exists.

The existence of the continuos selection of Φ was the main novelty in the proof of Antosiewicz and Cellina. The main idea was to interpolate continuously between a finite number of integrable functions by a method of "cutting and piecing": perturb a function cutting its graph on a measurable set piecing in this set a corresponding part of the graph of another function.

Decomposable sets

A set closed with respect to the cutting and piecing operation is called a decomposable set.

Definition 75 *$K \subset L^1(I, \mathbb{R}^n)$ is called decomposable if*

$$u\chi_A + v\chi_{I\setminus A} \in K$$

for all u, $v \in K$ and for all $A \subset I$ Lebesgue measurable.

Decomposable sets: substitute for convex sets

Fryszkowski (1981): any lower semicontinuous multifunction with decomposable values $\Phi(t, x)$ defined on a compact subset admits continuous selection.

Generalize Michael theorem and has been generalized by Bressan-Colombo (1988) (avoided compactness assumption). Results concerning continuous selections from solution sets: Cellina, Fryszkowski-Rzezuchowski-Colombo-S., Bressan-Cellina-Fryszkowski)

Selections method

For $F : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ we want to solve the problem

$$x' \in F(t, x), x(0) = x_0 \tag{132}$$

by the following method:

(i) find $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(t, x) \in F(t, x) \;\; \forall (t, x) \in [0, T] \times \mathbb{R}^n.$$

(ii) solve the Cauchy problem

$$x' = f(t, x), x(0) = x_0 \tag{133}$$

Any solution of (133) is also a solution of (132).

If F is l.s.c., closed convex valued then a continuos selection f from F exists by Michael theorem and solutions to the problem (133) exist by Peano's theorem.

If F is not convex valued, continuos selections from F may not exists, and so (i) cannot be realized if we want f continuous. Moreover, if f is not continuous then (133) may have not solutions.

Directional continuity

A property of $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ weaker than continuity, stronger than measurability, strong enough to guarantee the existence of solutions to (133). Let $M > 0$ and Γ^M be the cone

$$\Gamma^M = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \|x\| \leq Mt\}.$$

Definition 76 *A function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is called Γ^M - continuous if for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and for each sequence (t_n, x_n) converging to (t, x) and such that $(t_n - t, x_n - x) \in \Gamma^M$ for all $n \in N$, one has that*

$$lim_{n\to\infty} f(t_n, x_n) = f(t, x).$$

$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is called directionally continuous if it is Γ^M - continuous for some $M > 0$.

A. Pucci (1971) and Bressan (1988) proved that (133) admits Carathéodory solutions if f is directionally continuous.

Bressan (1988): for every $M > 0$, every l.s.c., closed values multifunction admits a Γ^M - continuous selection.

Bressan(1990): If $F : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a l.s.c., bounded, compact valued then (133) admits a solution on $[0, T]$.

Baire category method

The Baire category method, was introduced by De Blasi and Pianigiani (1985) and start from the Cellina's remark that the set of solutions of the Cauchy problem

$$x' \in \{-1, 1\}, x(0) = 0 \tag{134}$$

is a G_δ dense subset (i.e., a countable intersection of open dense subsets) of the set of solutions of the Cauchy problem

$$x' \in [-1, 1], x(t_0) = 0. \tag{135}$$

De Blasi and Pianigiani consider the problems

$$x' \in F(t, x), x(0) = x_0, \tag{136}$$

$$x' \in \partial F(t, x), x(t_0) = x_0 \tag{137}$$

where F is a continuous multivalued map, defined on an open set of $\mathbb{R} \times E$, with closed, convex bounded values having nonempty interior in a reflexive Banach space E, and $\partial F(t, x)$ denotes the boundary of $F(t, x)$.

Denote $\mathcal{M}_F$ and $\mathcal{M}_{\partial F}$ the set of solutions of (136) and (137), respectively.

Show that $\mathcal{M}_F$ is nonempty and that with the uniformly convergence metric, it is a complete metric space.

Show that $\mathcal{M}_{\partial F}$ is the intersection of a countable family of open dense subsets of $\mathcal{M}_F$, hence it is a G_δ dense subset of $\mathcal{M}_F$, hence it is nonempty and problem (137) admits solutions.

Comments

In a review of Deimling's book on Differential inclusions (published in Bull. Amer. Math. Soc., 1995), Cellina says: "differential inclusions have been a formidable gymnasium for the creation of ideas to treat non convex problems, but this instrument, mathematically challenging and stimulating as it had been, was essentially of no use in describing the real word.

This strong opinion has been changed and the reason is given by two examples that we will discuss: Fermat principle and the Brachystocrone problem.

Fermat (around 1650) stated the well known principle that the light, to pass from a point to a second point in space, through a medium where the speed of the light is (possibly) variable, among all the possible paths that join the two points, follows the path that minimizes the time.

The problem of the Brachystocrone: find the path, leading from a point P_1 to a point P_2 such that an object, subject to gravity only, falling along this path, would reach the point P_2 in minimum time.

Fermat principle

Fermat's principle predates of thirty years the celebrated Brachystocrone Problem of Jakob Bernoulli, and, remarkably, its aim is to explain a physical phenomenon that is discontinuous: Fermat's aim was to characterize the path followed by light in passing through two media with different speed coefficients, as water and air, and in particular to explain why a stick, partially sunk under water, seems to be broken to an observer.

We can model the problem in the following way: given two points P_1 and P_2 and the speed $\rho(x)$ of light at the point x, prove that among all the solutions to the differential inclusion

$$x'(t) \in \rho(x(t))\partial B \text{ or, equivalently, } \|x'(t)\| = \rho(x(t)).$$

joining P_1 and P_2 (the virtual trajectories followed by the light to travel from P_1 to P_2) there is a solution that travels from P_1 to P_2 in minimum time.

It is clear that we would like to prove this theorem under conditions on the function that allow it to be discontinuous , otherwise we do not prove what Fermat had in mind to prove.

Although at the beginning of the Calculus of Variations, this problem was modeled as the minimization of an integral functional, it is today recognized as being a typical minimum time problem for solutions to differential inclusions, and it happens that the differential inclusion involved is non- convex valued.

The minimization of functions where some non-convexity is involved is difficult, since the basic argument used in the Calculus of Variations breaks down.

Again the model present a lack of convexity, so that it is not lower semicontinuous, to use the standard language of the Calculus of Variations.

Still, the existence of a solution for the problem follows along the same lines as for Fermat's Principle.

Some open questions

- Investigate new existence results for solutions to non-convex differential inclusions and to the minimization of functionals constrained by non-convex differential inclusions.

- In the paper A. Cellina, A. Ferriero, and E. M. Marchini. On the existence of solutions to a class of minimum time problems and applications to Fermat's principle and to the Brachystocrone, published on Systems Control Lett., 55:119–123, 2006 a technique for proving existence of solutions for a class of minimum time problems, subject to differential inclusions with non-convex right hand side., is presented.

- However, besides the minimum time problem, other problems on a variable interval of time exist in the literature, mainly the optimal control literature; for instance, a purpose could be to reach the final point not necessarily in minimum time, but minimizing a functional of the kind

$$\int_0^t [1 + L(x(s), x'(s))] \, ds$$

where the term L takes into account the fact that the stress on the system has to be penalized.

- A similar problem is the attempt of proving existence results of solutions to differential inclusions with upper semicontinuous right hand side, by passing to the limit in the approximate solutions, not weakly, as it is customarily done, but strongly in the derivatives, following ideas in Cellina-Monti-Spadoni and Visintin.

- Study the existence of solutions depending continuously on affine boundary data for gradient inclusions Develop an existence theory for gradient inclusions with convex and nonconvex right-hand side. By the methods of upper and lower solutions (Goncharov-Ornelas, 2006) obtain extremal solutions and their density in the set of solutions of the convexified problem.

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